JIGSAW: Joint Inhomogeneity estimation via Global Segment Assembly for Water-fat separation

Wenmiao Lu and Yi Lu

Abstract—Water-fat separation in magnetic resonance imaging (MRI) is of great clinical importance, and the key to uniform water-fat separation lies in field map estimation. This work deals with three-point field map estimation, in which water and fat are modelled as two single-peak spectral lines, and field inhomogeneities shift the spectrum by an unknown amount.

Due to the simplified spectrum modelling, there exists inherent ambiguity in forming field maps from multiple locally feasible field map values at each pixel. To resolve such ambiguity, spatial smoothness of field maps has been incorporated as a constraint of an optimization problem. However, there are two issues: The optimization problem is computationally intractable and even when it is solved exactly, it does not always separate water and fat images. Hence, robust field map estimation remains challenging in many clinically important imaging scenarios.

This paper proposes a novel field map estimation technique called JIGSAW. It extends a loopy belief propagation (BP) algorithm to obtain an approximate solution to the optimization problem. The solution produces locally smooth segments and avoids error propagation associated with greedy methods. The locally smooth segments are then assembled into a globally consistent field map by exploiting the periodicity of the feasible field map values. In vivo results demonstrate that JIGSAW outperforms existing techniques and produces correct water-fat separation in challenging imaging scenarios.

Index Terms—belief propagation, water-fat separation, field map estimation, MRI, chemical-shift imaging

I. INTRODUCTION

Uniform water-fat separation in magnetic resonance imaging (MRI) is required in many important applications, such as suppressing overly bright fat signals to reveal underlying pathology and quantifying the distribution of adipose tissue in fatty liver diseases [1]. As water-fat separation in MRI is based on chemical-shift (∼3.5 ppm) induced phase difference between water and fat signals [2], robust field map estimation is crucial to ensure complete removal of the extra phase factors introduced by field inhomogeneities. The failure of field map estimation results in fragmented water and fat swaps in separated images. In this paper, we focus on three-point field map estimation, of which the inputs are three echoes acquired at uniformly spaced echo times (TE1, TE2, TE3), and the echo spacing ∆TE = TE2 – TE1 = TE3 – TE2) [3]–[5]. Three-point field map estimation for water-fat separation is gaining great interests mainly for the flexibility in prescribing imaging parameters and the ability to obtain optimal signal-to-noise ratio (SNR) of separation results [4].

The signal model commonly used in three-point water-fat separation is

\[ S_i = (W + F e^{2 \pi i \Delta f T E_i}) e^{2 \pi i \psi T E_i}, \]  

(1)

where the received signal \( S_i \) at the \( i^{th} \) echo time consists of water \( W \) and fat \( F \) signals, which has the phase difference induced by the chemical shift \( \Delta f \) (∼−210 Hz at 1.5 Tesla). In Eq. 1, water and fat are modelled as two spectral lines separated by the chemical shift \( \Delta f \). \( \psi \) is field inhomogeneity to be estimated, which shifts the spectrum by an unknown amount. The field map estimation essentially reverses the unknown shift of the spectrum, and the water-fat separation reduces to finding the magnitudes of the water and fat spectral lines.

The signal model in Eq. 1 ignores multiple spectral peaks of fat and \( T_2^* \) decays of chemical species, hence enabling the spectroscopic imaging of water and fat with the acquisition of merely three echoes. However, the simplified spectrum modelling leads to inherent ambiguity in resolving field inhomogeneity map [6]–[8]. When water and fat are modelled as two single-peak spectral lines, a pixel containing only water subject to field inhomogeneity \( \psi = 0 \) cannot be differentiated from one containing only fat (assume \( W = F \)) but subject to field inhomogeneity \( \psi = -\Delta f \). Therefore, in the presence of noise, the identities of the separation results cannot be determined for most pixels, which are dominant with one chemical species: the dominant spectral line can align with either water or fat.

In this work we define feasible field map values as the estimates of \( \psi \) that aligns the dominant spectral line with one chemical species such that the acquired three echoes have a good fit with the signal model in the least-squares sense. The feasible field map values at each pixel separate its water and fat signals without revealing the identities of the separation results. We explain how to locate feasible field map values by identifying the local minima of a least-squares cost function in Section II.

By viewing the uniform echo-spacing \( \Delta TE \) as the sampling interval along the temporal axis, the reciprocal of the echo-spacing, \( 1/\Delta TE \), corresponds to a spectral field-of-view, \( FOV_S \). This gives rise to multiple feasible field map values that separates water and fat at each individual pixel. When more than one \( FOV_S \) need to be considered for the complete coverage of possible field inhomogeneities, the set of feasible field map values is periodic with a period equal to \( FOV_S \). The periodic copies of a field map value produces the same water-fat separation result. However, pixels dominant with one
chemical species have two feasible field map values within one FOV, which produce two different separation results that correspond to a swap of the water and fat components.

A. Previous Approaches

As there exists inherent ambiguity in forming field maps from multiple feasible values at each pixel to obtain uniform water-fat separation, enormous amount of research effort has been devoted to resolving the ambiguity in field map estimation. To that end, many approaches have been proposed to incorporate the constraint of spatial smoothness into field map estimation. The constraint complies with the physical reality that in general field inhomogeneities vary smoothly over field-of-view (FOV).

The smoothness constraint can be formulated as follows. Let \( F_p \) denote the set of feasible field map values at pixel \( p \). These are field map values that separate water and fat signals for the corresponding pixel. Let \( \psi_p \) denote the field map value at pixel \( p \). The smoothness constraint gives the following optimization problem over the set of feasible values:

\[
\min \sum_p \sum_{q \in \mathcal{N}(p) \cap F_p} d(\psi_p, \psi_q),
\]

where \( \mathcal{N}(p) \) is the set of neighboring pixels to pixel \( p \) and \( d(\cdot) \) is a non-negative cost function of the difference between neighboring field map values.

Previous approaches to the optimization problem in Eq. 2 fall into two categories: One category of methods treats the smoothness constraint implicitly, and the other treats it explicitly by incorporating the constraint into the objective function to be optimized. The former category includes region growing [8] and region unwrapping [6] methods which implicitly deal with the smoothness constraint. The region growing method starts from one pixel and grows the region to include neighboring pixels so as to greedily maximize smoothness. The region unwrapping method divides the field map into relatively smooth regions using pre-defined thresholds and unwraps the regions to maximize smoothness across the boundaries. Both methods work well when aliasing is not severe: The region growing method assumes that most field map values are within the spectral FOV centered at 0 Hz. However, the greedy nature of the method can result in error propagation when spectral aliasing leads to the wrong selection of field map values. The region unwrapping method relies on the assumption that there exist large regions whose field map values fall in the same spectral FOV. However, this assumption breaks down in the presence of severe spectral aliasing, which results in a large number of regions to be unwrapped, and hence makes the method difficult to scale.

VARPRO [9, 10] treats the smoothness constraint explicitly and incorporates it as part of an objective function. The problem formulation in VARPRO is based on the Markov random field (MRF) framework. There are two parts in the objective function formulated in VARPRO: One is the data cost that measures the goodness of fit at an individual pixel; the other is the smoothness cost that measures the similarity of field map values at neighboring pixels. The optimization problem is tackled by iterated conditional modes (ICM) algorithm in [9] and graph cut algorithm in [10], respectively. The regularization parameter, which is often empirically chosen to combine the two parts in the objective function, determines the degree of smoothing in the resultant field maps. Over-smoothing of field maps is highly undesirable, especially when resolving rapid changes in field inhomogeneities, as field map values in a small neighborhood can have substantial variation. To avoid over-smoothing of field maps, the regularization parameter needs to be chosen such that the smoothness cost is to be optimized on the set of feasible field map values minimizing the data cost. In this case, the problem is reduced to Eq. 2, and is a challenging combinatorial optimization problem.

B. Difficulties in Field Map Estimation

The smoothness constraint summarized in Eq. 2 is essential to robust field map estimation. Nevertheless, there are difficulties associated with Eq. 2 in the two aspects:

1. Firstly, exact solution to the combinatorial optimization problem in Eq. 2 is often computationally intractable. For instance, with

\[
d(\psi_p, \psi_q) = (\psi_p - \psi_q)^2,
\]

a norm commonly used in previous methods, we show in the appendix that the problem is as hard as finding the ground state of a subclass of spin glass models with magnetic field, for which no polynomial time solution is known. In fact, the class of problems of finding the ground state of spin glass models with magnetic field is NP-hard [11].

2. Secondly, the exact solution to the optimization problem in Eq. 2, even when identified, not necessarily provides the correct field map. This is because there exist two feasible field map values in one spectral FOV for most pixels, one of which corresponds to the correct field map and the other to the alternative field map. The difference between these two values depends on water-fat ratio, echo times, and acquisition noise. The swapped field map can be locally smoother than the true field map; hence, the solution to the optimization problem in Eq. 2 could erroneously include the swapped field map, instead of the true field map.

C. Our Approach

To address the aforementioned difficulties, we propose a novel field map estimation technique called JIGSAW, which consists of two steps. The first step deals with the intractability of the optimization problem by obtaining an approximate solution with a low complexity algorithm, and the second step deals with the ambiguity due to the alternative field maps by introducing a new “consistency” measure between neighboring field map values.

Step 1: JIGSAW locates feasible field map values at each pixel and obtains an approximate solution to (2) using loopy belief propagation (BP) with decimation. The cost function in
Eq. 3 is used in this step. Note that the aim is not to solve the optimization problem exactly, but to produce locally smooth segments for global assembly in the next step.

Remark 1. The locally smooth segments produced by BP with decimation is different from those in the region unwrapping method. The region unwrapping method only identifies smooth regions that fall into the same spectral FOV, while BP with decimation identifies smooth segments across multiple spectral FOVs. Hence, the region unwrapping method only works with field map values modulo the FOV width. Depending on the width of spectral FOV, the number of segments produced by the region unwrapping method can be much larger than that produced by BP with decimation.

Step 2: We introduce the following “consistency” measure. Field map values \( \psi_p \) and \( \psi_q \) are considered “consistent” if and only if

\[
|\psi_p - \psi_q| \leq \max\left( \min_{j \in F_p} |\psi_p - \psi_j|, \min_{j \in F_q} |\psi_j - \psi_q| \right),
\]

where \( F_p \) and \( F_q \) denote the feasible sets of field map values at the neighboring pixels \( p \) and \( q \) respectively. JIGSAW assembles the locally smooth segments into a globally consistent field map with the “swap” move, which shifts the field map value for each pixel in the segment to a neighboring value in \( F_p \) so as to minimize inconsistent pairs across the segment boundaries.

Remark 2. Step 2 is effectively solving the following optimization problem:

\[
\min_{(p,q) \in E} J(\psi_p, \psi_q),
\]

where

\[
J(\psi_p, \psi_q) = \max( \min_{j \in F_p} |\psi_p - \psi_j|, \min_{j \in F_q} |\psi_j - \psi_q| ) - \sum_{(p,q) \in E} \left( |\psi_p - \psi_q| - \max( \min_{j \in F_p} |\psi_p - \psi_j|, \min_{j \in F_q} |\psi_j - \psi_q| ) \right).
\]

Note that the cost function \( J(\psi_p, \psi_q) \) does not favor small absolute variations between neighboring pixels, hence assigning high costs to inconsistent pairs across the segment boundaries. This cost function is not used in BP because it does not differentiate between a true field map and its alternative copy, hence deterting convergence of the message passing algorithm.

Remark 3. The swap move exploits the periodicity of the feasible field map values, and the fact that a smooth segment in the alternative field map will remain smooth in the true field map after the swap move. Intuitively, BP with decimation obtains locally smooth segments where the field map is varying slowly and easy to identify, while the swap move resolves boundaries at which the field map changes abruptly.

Section II describes the matrix formulation of the signal model and the concept of spectral field-of-view (\( FOV_s \)). Section III presents the proposed JIGSAW technique and its extension to 3D image data. The details of the MRI experiments are described in Section IV. We present in-vivo results from studies of ankle, liver and breast in Section V and discuss the results in Section VI.

II. Preliminary

A. Signal Model of Three-point Field Map Estimation

Let \( S_t \) denote the images acquired at the \( t \)th echo times, \( \tau_{TEi} \). When the echo times are much shorter than the \( T_2^* \)s of the tissues, we can ignore the magnitude difference due to \( T_2^* \) decay between acquisitions. The signal model of three-point field map estimation can be concisely summarized in the following matrix representation:

\[
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix} = \Psi(\hat{\psi})
\begin{pmatrix}
\frac{e^{12\pi\Delta\tau_{TE1}}}{A} & 1 & \frac{e^{12\pi\Delta\tau_{TE2}}}{W} \\
1 & \frac{e^{12\pi\Delta\tau_{TE2}}}{A} & 1 \\
\frac{e^{12\pi\Delta\tau_{TE3}}}{W} & 1 & \frac{e^{12\pi\Delta\tau_{TE3}}}{A}
\end{pmatrix},
\]

where \( W \) denotes the “chemical shift matrix”, and \( \Psi \) “field map matrix”, \( A \) “chemical shift matrix”, and \( \Gamma \) “signal vector.” The field map matrix \( \Psi \), which is determined by field map value \( \psi \), is a diagonal matrix with the following diagonal entries:

\[
\begin{pmatrix}
\frac{e^{12\pi\Delta\tau_{TE1}}}{A} & 1 & \frac{e^{12\pi\Delta\tau_{TE2}}}{W} \\
1 & \frac{e^{12\pi\Delta\tau_{TE2}}}{A} & 1 \\
\frac{e^{12\pi\Delta\tau_{TE3}}}{W} & 1 & \frac{e^{12\pi\Delta\tau_{TE3}}}{A}
\end{pmatrix}.
\]

Given an estimate of the field map value \( \hat{\psi} \) at a pixel \((x, y)\), its signal vector can be found optimally in a least-squares sense by solving

\[
\hat{\Gamma}(x, y) = A^\dagger \Psi(\hat{\psi}(x, y))^{-1} S(x, y),
\]

where \( A^\dagger = (A^\top A)^{-1} A^\top \).

B. Spectral Field-of-View

The least-squares estimation error of the signal vector \( \hat{\Gamma}(x, y) \) in Eq. 7 can be defined as a cost function \( J \) for the estimate of the field map value, \( \hat{\psi} \). The cost function is given by

\[
|J(\hat{\psi})| = ||(AA^\dagger - I)\Psi(\hat{\psi})^{-1} S||.
\]

The lower the least-squares error \( J \), the better the corresponding estimated field map value \( \hat{\psi} \). Figure 1 shows a typical plot of the cost function, which has two prominent characteristics—it is periodic and has multiple local minima. The fundamental period of the cost function is equal to the reciprocal of the echo-spacing (\( \Delta TE \)). Figure 1 shows the spectral field-of-view (\( FOV_s \)). It has been formally shown by Jacob et al. [6] that the number of feasible field map values inside the \( FOV_s \) is equal to 2 for most pixels which are dominant with either water or fat. In the noiseless scenario where a pixel contains only water or fat, the difference between two local minima inside \( FOV_s \) is \( \Delta f / \Delta TE \).

When the spectral FOV is not sufficient to encompass field map values, this leads to the aliasing of field map values in \( FOV_s \). Spectral aliasing exists in many important imaging scenarios, especially due to long echo spacings or high field strengths, and makes field map estimation difficult. For example, local greedy algorithms (e.g., region-growing algorithms) can erroneously select aliased copies of field map values, which nevertheless are not globally optimal choices and result in fragmented swaps of water and fat.
We denote the two field map values as the following two steps: Global Segment Assembly for Water-fat separation, which consists of the following two steps: 1. BP-guided decimation, which uses sum-product belief propagation (BP) to pass soft-decision messages among pixels and jointly estimates the most likely field map value for each pixel. Operating on the BP results, the “decimation” move fixes a segment with consistent field map values, referred to as a decimated set. Subsequently, the boundary values of the decimated set impose soft prior weights on the neighboring pixels.

### III. METHODS

In this section we describe the proposed field map estimation technique, JIGSAW (Joint Inhomogeneity estimation via Global Segment Assembly for Water-fat separation), which consists of the following two steps:

1. **BP-guided decimation**: which uses sum-product belief propagation (BP) to pass soft-decision messages among pixels and jointly estimates the most likely field map value for each pixel. Operating on the BP results, the “decimation” move fixes a segment with consistent field map values, referred to as a decimated set. Subsequently, the boundary values of the decimated set impose soft prior weights on the neighboring pixels.

2. **“Swap” move**: which shifts each field map value in a segment to its adjacent value in the feasible set, so as to minimize the number of inconsistent pairs across segment boundaries.

Note that the **decimated set** refers to the set of pixels whose field map values are fixed at a certain stage of the algorithm. A foreground mask is constructed to include the pixels with magnitude greater than the tenth percentile. The pixel at the centroid of the foreground mask serves as the initial decimated set. The decimated set is enlarged at each step until the field map values at all pixels are determined.

In the following we elaborate on the 2D JIGSAW algorithm in Section III-A and its 3D extension in Section III-B to cater for 3D data sets. The 3D extension of JIGSAW treats a completed 2D field map as the initial decimated set in 3D space such that the soft prior weight imposed by the 2D field map facilitates the field map estimation in its neighboring slice.

#### A. 2D JIGSAW

We describe the basic component of the algorithm, belief propagation, followed by an explanation of the “decimate” and “swap” moves. The 2D JIGSAW technique is outlined in Exhibit 1.

**Exhibit 1**: 2D JIGSAW

1. **Initialize the decimated set**
2. Fix the field map value to be \((\arg \min_{\psi_p \in F_p} |\psi_p|)\) for the pixel \(p\) at the centroid of the foreground mask
3. **Repeat till the decimated set includes all pixels**
4. Run sum-product BP with soft priors from current decimated set;
5. Enlarge the decimated set by including smooth segments of consistent field map values and with consistent boundary to the current decimated set;
6. If inconsistent boundaries remain after running BP
7. Identify a smooth segment with inconsistent boundaries from the current decimated set;
8. Swap the segment to minimize number of inconsistent pairs across segment boundaries;
9. Enlarge the decimated set by including the swapped segment

**Remark 4.** The ”swap” move is executed sequentially on smooth segments in Exhibit 1. Our implementation also includes a final step to check the consistency among all segments by considering the consistency cost of all combinations of segments.

1) **Sum-Product Belief Propagation**: The sum-product belief propagation [12] is an iterative message-passing algorithm on a graphical model where messages are passed from a pixel to all its neighbors on the graph. It is known that on a

1The starting pixel is selected under the assumptions that with reasonably good shimming, the central region in the FOV is close to on resonance condition. In addition, the starting pixel should have sufficiently high SNR such that the selected value of the starting pixel exists inside a smooth-varying field map that leads to uniform water-fat separation.

2Sum-product belief propagation is referred to as BP in the following where there is no ambiguity.
tree structure graph, or graph with a single loop, BP computes the correct marginal probability for the vector of values at each node [12]. For the field map estimation problem in this work, loopy BP [13] is used on a 2D image grid, where each pixel has four neighboring pixels. During an iteration, a message is passed from every pixel to its four neighbors, as illustrated in Fig. 2. Each message is a vector whose dimension is given by the number of feasible field map values at the receiving pixel, and each element of the message reflects the probability of the corresponding field map value.

For two neighboring pixels \( p \) and \( q \), \( N(p) \setminus q \) denotes the set of neighboring pixels to \( p \) other than \( q \). Let \( F_p \) and \( F_q \) denote the feasible sets of field map values at the pixels \( p \) and \( q \) respectively. Eq. 9 shows the computation of one element of the message going from \( p \) to \( q \) at the \( t \)-th iteration. In particular, it conveys the probability that the pixel \( q \) takes the value \( \psi_q \in F_q 

\[
m_{p \rightarrow q}^t(\psi_q) = \sum_{\psi_p \in F_p} \left( V_{pq} \prod_{s \in N(p) \setminus q} m_{s \rightarrow p}^{t-1}(\psi_p) \right) \tag{9}
\]

where

\[
V_{pq} = \exp\left(- (\psi_p - \psi_q)^2 / \sigma^2 \right) \tag{10}
\]

denotes the interaction potential between two field map values \( \psi_p \) and \( \psi_q \). The interaction potential takes the form of a Gaussian distribution, and the magnitude of the standard deviation \( \sigma \) can be conveniently set to be the spectral FOV, \( FOV_\sigma \). Note that the value of \( \sigma \) is largely chosen for numerical precision issues: the interactive potential \( V_{pq} \) captures a normalized difference between \( \psi_p \) and \( \psi_q \).

Each element of the message conveys the likelihood of the corresponding field map value, summed over the probability of each field map value for the sending pixel (hence the name sum-product). The interpretation of the message in Eq. 9 is as follows: the terms in the product computes the probability of node \( p \) taking field map value \( \psi_p \), based on beliefs from neighboring nodes other than \( q \). For each field map value \( \psi_p \), the probability of \( \psi_q \) at pixel \( q \) is computed via the interaction potential \( V_{pq} \). The overall probability of \( \psi_q \) is obtained by summing over the distribution of \( \psi_p \).

In each iteration, the messages received at a pixel are updated by taking into account the inference from its neighbors, which summarizes the current belief of the most probable field map values. In practice, only a few iterations are required to pass the messages across the entire 2D image grid. In our implementation, after \( T = 20 \) iterations, a belief vector \( b_q \) is computed for the pixel \( q \). The element of \( b_q \) corresponding to the value \( \psi_q \) is given by

\[
b_q(\psi_q) = \prod_{p \in N(q)} m_{p \rightarrow q}^T(\psi_q). 
\]

Subsequently, for the pixel \( q \), the field map value \( \psi_q^* \) that corresponds to the largest element of the belief vector \( b_q \) (i.e., the maximum marginal probability) is selected.

As image grids are fully connected with loops, it is well known that loopy BP algorithm does not yield exact optimal solution [14]. Direct application of the BP algorithm to field map estimation only produces piecewise smooth segments. To obtain a globally smooth-varying field map, we assemble those segments with the following two moves, namely decimation move and swap move.

2) Decimation Move: Decimation refers to the fixation of field map values at one or more pixels. By decimating successively, and running BP with prior beliefs from the decimated set, the algorithm converges to one specific solution of the problem. The first pixel, which is chosen at the centroid of the foreground mask, serves as the initial decimated set. The message from a decimated pixel \( p \) whose field map value is fixed to \( \psi_p \) to its neighboring “free” (not decimated) pixel \( q \) is

\[
m_{p \rightarrow q}^t(\psi_q) = V_{pq} \tag{11}
\]

for all \( t \). Only decimated pixels on the boundary of the decimated set sends messages. The messages serve as initial potential to guide the selection of field map values at the rest of the pixels. After the new field map values are chosen for the rest of the pixels based on the beliefs from the BP algorithm, the decimated set is enlarged to include pixels that are “consistent” with their neighbors, where the consistency condition is defined in Eq. 4.

The decimation move aims to ensure that the field map values included in the decimated set contain as few inconsistent pairs as possible. Due to the loopy nature of the BP algorithm, a large neighborhood of pixels can select field map values that are consistent among themselves, but ignore the guidance from the boundary values of the decimated set. This results in smooth segments bordering the decimated set, but retaining inconsistent boundaries between them, as shown in Fig. 3. In this case, a smooth segment denoted by \( S \) is formed at a level different from that of the decimated set (encircled by the bold line). To remedy such problems, we propose the swap move based on the following observation: the ground truth field map, with values at all pixels shifted to the adjacent value in the respective feasible sets, also separates water and fat signals consistently.

3) Swap Move: The swap move exploits the characteristics of feasible field map values for most pixels dominant with single species, that is, adjacent values in a feasible set lead to the swap of separated water and fat signals. A smooth segment, which has inconsistent boundaries with the decimated set, contains important information for the consistent solution. In fact, the consistent solution can be obtained by shifting each
field map value in the entire segment to its adjacent value in the feasible sets simultaneously.

The swap move works as follows: Let $S$ be a smooth segment within which the pixels are consistent among themselves; i.e., for any two neighboring pixels $p$ and $q$, $|\psi_p - \psi_q|$ satisfies the smoothness constraint in Eq. 4. Let $\psi_S$ denote the field map restricted to the segment $S$. Suppose for each pixel the values in its feasible set is ordered from the smallest to the largest, and $t_p$ denotes the index of the current value for $\psi_p$ in the feasible set $F_p$. Define the operator $\Gamma^k$ on the restricted field map $\psi_S$ as the component-wise operator where

$$\Gamma^k \psi_p = F_p(t_p + k), \forall p \in S,$$

where a positive $k$ shifts each field map value in $S$ to the $k$-th value after it in the feasible set, and a negative $k$ shifts each field map value to the $|k|$-th value before it in the set. The swap move locates the set of values under the operator $\Gamma^k$ that produces the smoothest boundary to the decimated set $D$:

$$\psi_S^j = \Gamma^j \psi_S$$

where

$$j = \arg \min_k \sum_{p \in D, q \in S, q \in N(p)} \mathbb{I}(\Gamma^k \psi_q, \psi_p),$$

and $\mathbb{I}(\Gamma^k \psi_q, \psi_p)$ is the same as that in Eq. 5. Hence, the shift results in $\psi_S^j$ (the new field map restricted to the segment $S$), which minimizes inconsistent pairs across segment boundaries.

### B. 3D JIGSAW

As field inhomogeneities should also be smooth-varying within the same tissue along the slice-select axis, one possible extension of JIGSAW to 3D is direct application of BP message passing on a 3D image grid. However, BP message passing on a 3D image grid involves much higher computational complexity and memory requirement than its 2D counterpart. This is because a 3D image grid has $N_s$ times more pixels than a 2D image grid ($N_s$ is the number of slices), and each pixel on a 3D image grid has more neighboring pixels. For computational efficiency, in the work we extend JIGSAW to 3D by treating a 2D field map as an initial decimated set in 3D space.

Let $I_1$ be the 2D slice whose field map has been estimated using the 2D JIGSAW technique, and let $I_2$ be a neighboring slice whose field map is to be estimated. Let $p_1$ and $p_2$ denote two neighboring pixels along the third dimension, with $p_1$ and $p_2$ sharing the same 2D coordinate in slices $I_1$ and $I_2$. The BP message from pixel $p_2$ to a neighboring pixel $q_2$ in slice $I_2$ at $t$-th iteration is given by

$$m^t_{p_2 \rightarrow q_2}(\psi_{q_2}) = \sum_{\psi_{p_2} \in F_{p_2}} W_{p_2} \prod_{s \in N(p_2) \setminus q_2} m^{t-1}_{s \rightarrow p_2}(\psi_{p_2})$$

(12)

where

$$W_{p_2} = \exp(-(|\psi_{p_1} - \psi_{p_2}|^2 / \alpha^2)),$$

is the interaction potential along the third dimension between the two field map values $\psi_{p_1}$ and $\psi_{p_2}$. Since the value $\psi_{p_1}$ is fixed, $W_{p_2}$ can be viewed as a self-potential or prior weight on the feasible set $F_{p_2}$.

The computation of the element of the belief vector $b_{q_2}$ is modified to be

$$b_{q_2}(\psi_{q_2}) = W_{q_2} \prod_{p_2 \in N(q_2)} m^T_{p_2 \rightarrow q_2}(\psi_{q_2}),$$

which is the product of all incoming messages from 2D neighbors multiplying with the self potential due to the 3D neighbor. The field map of $I_2$ is simply chosen to be the value that maximizes the belief at each pixel. No decimation or swap move is needed as the self-potential due to the field map of $I_1$ supplies information from an additional dimension and greatly reduces ambiguity from the 2D problem. On the other hand, the soft nature of the self-potential well accounts for potentially large variation of inhomogeneities along the slice-select axis.

### IV. MRI EXPERIMENTS

Robust field map estimation becomes exceedingly difficult when the spectral field-of-view $FOV$ is not sufficient to encompass feasible field map values. Large field inhomogeneities occur when imaging at high field strengths, and imaging around anatomies of irregular shapes and complex air-tissue interfaces. The aliasing of feasible field map values impose great challenges on existing field map estimation techniques.

One clinically important scenario causing such spectral aliasing is associated with multi-echo gradient recalled echo (GRE) sequences, which acquire multiple echoes in a single repetition ($TR$). By removing extra radio frequency pulses and spoiler gradients, multi-echo GRE sequences greatly reduce the prolonged scan times incurred in three-point field map estimation, thus enabling many important applications that require shorter breath-hold and less motion artifacts. To achieve even higher SNR efficiency than the unipolar multi-echo GRE sequence, bipolar multi-echo GRE sequences have been developed by removing fly-back gradients [15]. Figure 4 shows the schematic diagrams of conventional imaging sequence, unipolar and bipolar multi-echo GRE sequences for three-point water-fat separation. It can be seen that the multi-echo
GRE sequences greatly reduce the scan times incurred to acquire three echoes. However, due to the packing of readout gradients, the echo-spacings in both unipolar and bipolar multi-echo GRE sequences cannot be arbitrarily reduced to avoid spectral aliasing.

To demonstrate the efficacy of JIGSAW in resolving field inhomogeneities in the presence of spectral aliasing, we implemented both unipolar and bipolar multi-echo GRE sequences at a GE 3 T scanner with gradients capable of 40 mT/m amplitude and 150 T/m/s slew rate (GE Healthcare, Milwaukee, WI). The multi-echo GRE sequences were used to scan three challenging anatomical regions, abdomen, ankle and breast under the local institutional review board. The echo times of these three studies were chosen to make the phase differences between water and fat signals be $2\pi/3 \pm k\pi$ ($k$ is an integer number), which yields optimal SNR of the separation results [16].

Water-fat separation of ankle scans at 3 T is challenging due to large field inhomogeneities from inadequate shim and/or susceptibility variations. The unipolar multi-echo GRE sequence was used to scan the ankle of a healthy volunteer. A quadrature extremity coil was used to scan the ankle of a healthy volunteer. An 8-channel Cardiac array coil was used with the following imaging parameters: $TE_{1,2,3} = 3.5, 6.7, 9.8$ ms, acquisition matrix $256 \times 144 \times 28$ with phase FOV ratio 0.75, $FOV = 40 \times 30 \times 4$ cm, receive bandwidth $\pm 41.67$ kHz, $TR = 14$ ms. The liver volume was imaged during a 42-second breath-hold. Note that the long echo-spacing $3.2$ ms might not be suitable for imaging patients with iron-loaded livers, which have substantially short $T_2^*$s. For those patients, the $T_2^*$ effects cannot be neglected, and should be incorporated into the signal model, as suggested by Yu et al. [17].

Using a similar imaging protocol to the liver study, the same bipolar multi-echo GRE sequence was used to scan the breast of a healthy volunteer with an 8-channel breast array coil and the following imaging parameters: $TE_{1,2,3} = 3.5, 6.7, 9.8$ ms, acquisition matrix $256 \times 128 \times 26$ with phase FOV ratio 0.75, $FOV = 40 \times 30 \times 4$ cm, receive bandwidth $\pm 41.67$ kHz, $TR = 14$ ms. The breast volume was imaged during a 35-second breath-hold.

The large field inhomogeneities together with the long echo-spacing in these studies cause severe spectral aliasing, which present great challenges to robust field map estimation. We compare the results obtained from the 3D JIGSAW with that from IDEAL with region growing [8] implemented on the GE scanner and one recently proposed multi-resolution field map estimation technique [7], which overcomes some limitations of previous region-growing algorithms by taking into account multiple feasible field map values at each pixel. While the removal of fly-back gradients leads to better acquisition efficiency, the alternating readout gradients in bipolar multi-echo sequences introduce additional phase variations that need to be carefully corrected prior to field map estimation [15]. As the on-line region-growing IDEAL cannot be used to process the images acquired using the bipolar multi-echo GRE sequence, for the liver and breast studies, we corrected the phase variations in bipolar acquisitions, and compare JIGSAW with the off-line multi-resolution field map estimation technique.

V. RESULTS

The experimental results reported in this section were obtained from applying 3D JIGSAW to the data sets of the in-vivo studies. Figure 5 shows the comparison of field maps and separated water/fat images of a sample slice of the ankle study, obtained using JIGSAW and region-growing IDEAL implemented on the GE scanner. Due to the rapid field inhomogeneity variation from -450 Hz to 400 Hz, the on-line region-growing IDEAL produces a highly fragmented field map, resulting in the failure of water-fat separation. In contrast, JIGSAW resolves the field inhomogeneities in this challenging scenario and uniformly separates water and fat.

Figure 6 presents the sample separation results and the estimated field map for the result comparison of the liver study. The difficulty with the liver study is the lack of signal in the air gaps. Hence, the field inhomogeneity suffers a discontinuity at the boundary of air gaps, and the lack of neighboring pixels makes it harder to enforce the smoothness constraint. However, JIGSAW managed to resolve the large field inhomogeneities and achieve uniform water-fat separation.

Figure 7 shows the separation results and field maps of the breast study. We present one sample slice for the result comparison. In this case, the multi-resolution estimation manages to trace the field variation for most regions, but fails
Fig. 5. Comparison of field maps and water-fat separation results of a sample slice in the ankle study scanned with a unipolar multi-echo sequence. Note that the region-growing IDEAL implemented on the GE scanner fails to trace the rapid change of field inhomogeneities, resulting in highly fragmented water and fat swaps. Two regions of the swapped water-fat separation in the IDEAL results are identified with arrows. In contrast, JIGSAW produces globally uniform water-fat separation results with the globally smoothly varying field map.

Fig. 6. Comparison of field maps and water-fat separation results of a sample slice of the liver study scanned with a bipolar multi-echo sequence. Due to the discontinuities in field inhomogeneities at the tissue-air interfaces and the lack of neighboring pixels at the boundary of air gaps, it is difficult for the multi-resolution field map estimation technique to globally enforce spatial smoothness constraint. JIGSAW does not suffer from the similar failure and achieves uniform water-fat separation.

Fig. 7. Comparison of field maps and water-fat separation results of a sample slice of the breast study scanned with a bipolar multi-echo sequence. The multi-resolution estimation manages to trace the field variation for most regions, but fail at the place where the field inhomogeneities change rapidly. This results in the swap of water and fat, as pointed by the arrows. In contrast, JIGSAW robustly traces the field variation, producing uniform water-fat separation across the entire FOV.

Robust field map estimation becomes very challenging in the presence of spectral aliasing of feasible field map values. In this work, JIGSAW provides an effective solution to address the difficulties in field map estimation caused by the spectral aliasing.

JIGSAW is resilient to the error propagation associated with greedy methods, which make hard decisions at each step to greedily minimize the smoothness cost. The BP-guided decimation in JIGSAW exchanges soft-decision messages across image grids, which convey the likelihoods of field map values by gathering locally summarized information from a large portion of the image. As a result, the BP-guided decimation leads to a configuration where neighboring field map values are close to each other, hence automatically producing piecewise smooth segments that can be assembled into a globally smooth-varying field map.

Many existing techniques explicitly impose the constraint of spatial smoothness by formulating the smoothness cost (Eq. 2) into a joint optimization problem. We have shown that it is difficult to solve such optimization problems exactly, and even the exact solution might not correspond to the correct field map. Instead of minimizing a specific cost function, JIGSAW provides an approximate solution to minimize the occurrence of inconsistent field map values. The consistency constraint in Eq. 4 is designed to capture the desirable property of spatial smoothness; that is, only abrupt changes, instead of all the
differences, between neighboring field map values should be minimized.

Another potential problem of the joint optimization is over-smoothing of field map associated with regularization parameters that combine the data fitting cost with the smoothness penalty. In contrast, JIGSAW de-couples water-fat separation at individual pixels and field map estimation at an image level. This ensures that the resulting field maps do not suffer from over-smoothing, hence maintaining the maximum likelihood water-fat separation obtained from locating the local minima of the least-squares cost function. This is important for the SNR optimality of the water-fat separation results.

2D JIGSAW has a complexity linear with the size of image grids. The main computation burden lies in the BP message passing across the image grid. Without attempting the exact minimization of the cost function for the spatial smoothness constraint of field maps, a small number of iterations (20 in our implementation) is needed to form locally smooth segments by exchanging the messages among neighboring pixels. On the other hand, the computational costs incurred by the swap move depend on the severity of field inhomogeneities in different studies. In general, large and rapid variation of field inhomogeneities necessitates the invocation of more swap moves to identify segments inconsistent with the decimated set. It typically takes about 30 seconds by 2D JIGSAW implemented in MATLAB running on a Linux machine equipped with Intel Core i5 750 Processor 2.66 GHz and 4 GB memory to produce a field map of matrix size $256 \times 256$.

3D JIGSAW achieves fast convergence by utilizing a completed 2D smooth varying field map as a prior weight to guide the estimation of neighboring field maps. As the initial 2D field map imposes strong initial weights on the soft-decision messages passed between the nodes in neighboring slices, the rest of the field maps can be obtained with few iterations of BP message passing without invoking the swap move. The main computation burden of 3D JIGSAW lies in obtaining a reliable initial 2D field map. To mitigate the possibility that errors in the first 2D field map propagate through 3D JIGSAW, 2D JIGSAW is applied to independently produce the field maps for a few slices close to the center of the 3D slab. In our implementation the field maps of three middle slices in the 3D slab are computed, among which the smoothest one is chosen as the initial 2D decimated set for the 3D JIGSAW. Once the initial 2D field map is selected, the field map propagation across the slice-select direction incurs little computational cost. It typically takes less than 3 seconds for the propagation across one slice of matrix size $256 \times 256$ on our machine.

The three-point signal model considered in this paper (Eq. 6) does not include $T_2^*$ decays and multiple spectral peaks of fat signals. While this simplified signal model suffices in many clinical applications, a detailed signal model that accounts for the multiple spectral peaks of fat [18] is necessary for other purposes, in particular fat quantification. In addition, Yu et al. [19] proposed to exploit the multiple spectral peaks of fat to estimate the likelihoods of water and fat existence in each pixel, which greatly improves the robustness of field map estimation. The calibration of the multiple spectral peaks of fat requires the acquisition of more echoes (e.g., 6 echoes). While JIGSAW was developed primarily for three-point water-fat separation, it is advantageous to extend JIGSAW to incorporate the multiple spectral peaks of fat. Specifically, the feasible field map values at each pixel can be assigned with a priori likelihoods derived from the detailed signal model, which serve as the initial node potential in the loopy BP. Such extension is expected to further improve the robustness and convergence speed of JIGSAW.

VII. CONCLUSION

We introduced a new estimation technique called JIGSAW that is designed to resolve ambiguity in field map estimation caused by the aliasing in spectral FOV. JIGSAW is potentially useful for water-fat imaging at high field strengths and emerging clinical applications involving water-fat separation with multi-echo imaging sequences. Initial experimental results show JIGSAW holds great promise in producing robust field map estimation in challenging magnetic resonance imaging scenarios. For the future work, we plan to validate the clinical utility of JIGSAW via more thorough clinical studies.

APPENDIX A

We show that the field map estimation problem is as hard as the problem of finding the ground state of a subclass of spin glass models with magnetic field.

**Definition 1:** $\text{Minset}_{pq}$. The $\text{Minset}_{pq}$ for an edge $(pq)$ in the grid is a set of non-negative real numbers constructed as follows. Let the number of feasible values for $q$ in each period be $k \in \{1, 2\}$. For each feasible value of $p$ in the period centered at zero, identify the $k$ feasible values of $q$ with the smallest distances. The values of the distances constitute $\text{Minset}_{pq}|p$.

Note that $\text{Minset}_{pq}|p$ and $\text{Minset}_{pq}|q$ have the same cardinality. The cardinality is 1 when both $p$ and $q$ have only one feasible value in each period, 2 when exactly one of them has two feasible values in each period and 4 when both of them have two feasible values in each period.

**Lemma 1:** Given an edge $(pq)$,

\[
\text{Minset}_{pq}|p = \text{Minset}_{pq}|q.
\]

Hence, we denote the set $\text{Minset}_{pq}$.

**Proof:** The proof is trivial for $|\text{Minset}_{pq}|p| = 1$ and $|\text{Minset}_{pq}|p| = 2$. We will show it for $|\text{Minset}_{pq}|p| = 4$.

Denote the two feasible values of $p$ in one period by $\psi_{p,1}$ and $\psi_{p,2}$, and denote those for $q$ by $\psi_{q,1}$ and $\psi_{q,2}$. Let $x_1 \leq x_2 \leq x_3 \leq x_4$ be the distances between $\psi_{p,1}$ and the four feasible values of $q$ closest to $\psi_{p,1}$. Due to the periodic nature of feasible values, $x_1$ and $x_3$ are distances between $\psi_{p,1}$ and two copies of the same feasible value in neighboring periods. w.l.o.g., let this feasible value be $\psi_{q,1}$. Similarly, let $y_1 \leq y_2 \leq y_3 \leq y_4$ be the corresponding four minimum distances for $\psi_{p,2}$. Then

\[
\text{Minset}_{pq}|p = \{x_1, x_2; y_1, y_2\}.
\]

Consider, w.l.o.g., the case where $y_1$ is the distance between $\psi_{p,2}$ and a copy of $\psi_{q,1}$, that is, $\psi_{q,1} + kFOV_s$, for some
and $y$ equivalent to showing with the energy function between the two copies of $\psi_{q,1}$, and two copies of $\psi_{p,1}$. The periodicity of feasible values indicate that there has to be a copy of $\psi_{p,2}$ between the two copies of $\psi_{p,1}$, and let the distance between $\psi_{q,1}$ and the copy of $\psi_{p,2}$ be $z$. Since $z \leq \max(x_1, x_3) = x_3$ and $y_1 \leq z \leq x_3$.

Similarly, we can show $x_1 \leq y_3$.

**Definition 2:** Admissible field map. A field map $\Psi$ is admissible if for every edge $(pq)$, $|\psi_p - \psi_q| \in \text{minset}_{pq}$.

**Remark.** All field maps we have observed, except those near metallic implants, are admissible. The field inhomogeneities are in general much smaller than the spectral FOV. The ambiguity in identifying the correct field map arises due to the alternative feasible values caused by aliasing.

Given a grid, for each node $p$, assign a set of real numbers $A_p$, with $|A_p| = 2$. Construct a subclass of spin glass model with magnetic field from the grid as follows. Denote the elements of $A_p$ by $A_{p,1}$ and $A_{p,2}$. On an edge $(pq)$, solve for $J_{pq}$, $h_{p \rightarrow q}$ and $h_{q \rightarrow p}$ from the following equations:

$$-J_{pq} + h_{p \rightarrow q} + h_{q \rightarrow p} = (A_{p,1} - A_{q,1})^2 + C$$

$$J_{pq} + h_{p \rightarrow q} - h_{q \rightarrow p} = (A_{p,1} - A_{q,2})^2 + C$$

$$J_{pq} - h_{p \rightarrow q} + h_{q \rightarrow p} = (A_{q,2} - A_{p,1})^2 + C$$

$$-J_{pq} - h_{p \rightarrow q} - h_{q \rightarrow p} = (A_{p,2} - A_{q,2})^2 + C,$$

where $C$ is a constant to be solved.

Let $h_p = \sum_{q \in N(p)} h_{q \rightarrow p}$. The spin glass model is defined with the energy function

$$H = -\sum J_{ij} S_i S_j + \sum h_i S_i,$$

with $S_i, S_j \in \{-1, 1\}$. We denote this subclass of spin glass models by $\Lambda$.

**Theorem 1:** Given an admissible solution, the combinatorial optimization problem in Eq. 2 with $d(\psi_p, \psi_q) = |\psi_p - \psi_q|^2$ is as hard as finding the ground state for the class of spin glass models $\Lambda$.

**Proof:** We prove the theorem by showing that for every spin glass model in $\Lambda$, we can construct a field map estimation problem with an admissible solution such that the solution to the latter will yield the ground state of the spin glass model.

Let $B_{pq}$ be the set of four values

$$B_{pq} = \{-J_{pq}, S_p S_q + h_{p \rightarrow q} S_p + h_{q \rightarrow p} S_q, S_p, S_q \in \{-1, 1\}\},$$

and

$$m_{pq} = \max \{b : b \in B_{pq}\}.$$

We can construct a field map estimation problem by letting $A_q$ be the two neighboring field map values at node $q$ and the period

$$FOV_q = m_{pq} + 1.$$

For each edge $(pq)$, and every element $b \in B_{pq}$, we have

$$FOV_q - b > b,$$

hence $\text{minset}_{pq} = B_{pq}$. Moreover, the sets $A_q$ are such that the distances between the values on any two nodes connected by an edge $(pq)$ fall in $\text{minset}_{pq}$.

It suffices to show that the solution to the constructed field map estimation problem will be admissible and takes value in the set $A_q$, since the ground state of the spin glass model will be identified by the one-to-one correspondence between $A_q$ and the spins $S_q$. Showing that the solution is admissible is straightforward since for any configuration that takes value outside $A_q$, we can replace each value outside $A_q$ by its copy in $A_q$ and obtain a solution with a smaller cost.

**References**


