MESSAGE PASSING FOR IN-VIVO FIELD MAP ESTIMATION IN MRI

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ABSTRACT
Robust field map estimation is important to many MRI applications, such as reconstruction with correction of susceptibility artifacts, MR-based temperature mapping, and water-fat separation. To enable in-vivo field map estimation with minimal scan times, multi-echo imaging sequences, which acquire multiple images in a single repetition, are gaining great interest. However, it has been observed that field map estimation becomes less reliable with multi-echo imaging sequences, especially at high field strengths and around challenging anatomies where good shimming cannot be obtained. In this paper, the field map estimation is shown to be a high-dimensional combinatorial optimization problem, which cannot be addressed by local greedy algorithms. This paper describes an effective approach based on message passing algorithm to globally approximate a solution with maximum a posterior (MAP) probability.

Index Terms— MRI, field inhomogeneity, in-vivo imaging, combinatorial optimization, message passing

1. INTRODUCTION
Recently, there is growing interest in acquiring in-vivo field maps with multi-echo imaging sequences, mainly motivated by time critical MR applications, such as MR-based temperature mapping and water-fat separation. To enable in-vivo field map estimation with minimal scan times, multi-echo imaging sequences, which acquire multiple images in a single repetition, are gaining great interest. However, it has been observed that field map estimation becomes less reliable with multi-echo imaging sequences, especially at high field strengths and around challenging anatomies where good shimming cannot be obtained. In this paper, the field map estimation is shown to be a high-dimensional combinatorial optimization problem, which cannot be addressed by local greedy algorithms. This paper describes an effective approach based on message passing algorithm to globally approximate a solution with maximum a posterior (MAP) probability.

Existing techniques for multi-point field map estimation include region growing [1], VARPO [2] and region unwrapping method [3] used in [4]. Region growing algorithms assume that most field map values are within the spectral FOV centered at 0, and large field map values can be extrapolated. VARPO optimizes a cost function that combines the goodness of fit of field map values at individual pixels and the smoothness cost among pixels. The region unwrapping method divides the field map values into pre-defined ranges and minimizes boundary difference among the regions formed thereby.

This paper describes an effective technique that performs robust field map estimation from multi-echo sequences using message passing algorithm. The message passing algorithm is based on belief propagation (BP) guided decimation that first used in statistical physics for combinatorial constraint satisfaction problems. By jointly determining field inhomogeneities with the BP algorithm, the proposed technique produces large segments of pixels within which field map values are consistent. The field map estimation problem is then reduced to the assembly of a few large segments, which can be easily achieved by considering field map values on the segmentation boundaries.

2. SIGNAL MODEL
Let $I_n$ denote the image acquired at the $n$-th echo time, $T_E_n$. For three-point methods, $n = 1, 2, 3$, as in [5]. A pixel at the location $(x, y)$ of the acquired image has complex signal values $I_n(x, y)$. Considering two most common chemical species in a human body, water and fat, we have the following signal model:

$$
\begin{pmatrix}
I_1(x, y) \\
I_2(x, y) \\
I_3(x, y)
\end{pmatrix}
= \begin{pmatrix}
\exp(i2\pi\psi(x,y)T_E_1) & 0 & 0 \\
0 & \exp(i2\pi\psi(x,y)T_E_2) & 0 \\
0 & 0 & \exp(i2\pi\psi(x,y)T_E_3)
\end{pmatrix}
\begin{pmatrix}
\Psi^*(\psi(x,y)) \\
\exp(i2\pi\Delta T E_1) \\
\exp(i2\pi\Delta T E_2) \\
\exp(i2\pi\Delta T E_3)
\end{pmatrix}
\begin{pmatrix}
W(x, y) \\
F(x, y)
\end{pmatrix},
$$
where we denote $S(x,y)$ the “echo signal vector”, $\Psi(\psi(x,y))$ the “field map matrix”, $A$ the “chemical shift matrix” ($\Delta f \approx -210$Hz at 1.5 T is the chemical shift between water and fat signals), and $\Gamma(x,y)$ the “signal vector” ($W(x,y)$ and $F(x,y)$ are complex water and fat signal values). Given an estimate of the field map value $\hat{\psi}(x,y)$ at $(x,y)$, the signal vector can be found optimally in a least-squares sense by solving

$$\hat{\Gamma}(x,y) = A^T \Psi(\hat{\psi}(x,y))^{-1} S(x,y),$$

where $A^T = (A^+ A)^{-1} A^+$ and $\Psi(\hat{\psi}(x,y))^{-1}$ is a diagonal matrix with diagonal entries:

$$e^{-j2\pi\psi(x,y)TE_1}, e^{-j2\pi\psi(x,y)TE_2}, e^{-j2\pi\psi(x,y)TE_3}.$$

### 3. PROBLEM FORMULATION

With reference to Eq. 1, a least-squares estimate of the signal vector $\hat{\Gamma}(x,y)$ is associated with an estimation error. For notational simplicity, the location index $(x,y)$ is dropped to reflect the dependence of $J$ and $\Psi$ on the estimated field map $\hat{\psi}$. The estimation error can be defined as a cost function $J$ for the estimate of the field map value, $\hat{\psi}$. The cost function is given by $J(\hat{\psi}) = \| (A^T A)^{-1} \Psi(\hat{\psi})^{-1} S \|$, which has multiple local minima and is periodic with fundamental period of $1/\Delta T$. Note that the period of the cost function is the same as the spectral FOV within which field inhomogeneities can be resolved without ambiguity. All the local minima can be identified by sampling the cost function at a reasonable interval (e.g., 1 Hz), followed by locating the field map value yielding lower costs than their immediate neighboring values. Each local minimum of the least-squares cost function provides a feasible field map value.

The common failure in field map estimation manifests in the form of fragments, which result from erroneous selection of the feasible field map values and their replicas outside the spectral FOV. The erroneous selection is often caused by the aliasing in the spectral FOV. When aliasing in the spectral FOV occurs, local greedy algorithms (e.g., region-growing), which select field map values only based on local neighboring pixels, often fail to produce globally smooth-varying field maps. Since each pixel has its own set of feasible field map values, forming a globally smoothly varying field map is a combinatorial problem: each pixel selects one feasible value such that the resultant field map does not have any abrupt jumps. To accommodate the scenarios with spectral aliasing, we propose the following smoothness constraint, which minimizes the number of inconsistent neighboring field map values.

The selected field map values $\psi_p$ and $\psi_q$ are “inconsistent” if

$$|\psi_p - \psi_q| > \max \left( \min_{\psi_j \in J_p} |\psi_p - \psi_j|, \min_{\psi_j \in J_q} |\psi_j - \psi_q| \right),$$

where $J_p$ and $J_q$ denote the feasible sets of field map values at the neighboring pixels $p$ and $q$, respectively. That is, two neighboring field map values are considered inconsistent when none of the two values are the closest to the other. The objective is to obtain reliable estimation of field maps with as few inconsistent pairs as possible.

### 4. ALGORITHM

Exhibit 1 outlines the proposed technique, which consists of the following three algorithmic components:

1. Sum-product belief propagation (BP) algorithm, which passes soft-decision messages among pixels and jointly estimates the most likely field map value for each pixel.
2. BP-guided decimation, which fixes a segment with consistent field map values. That is, for each pixel in the fixed segment, one field map value is selected from the feasible set and stays unchanged. Subsequently, the boundary values of the fixed segment are used to impose soft prior weights on the neighboring pixels.
3. “Swap” move, which shifts each field map value in a segment to its adjacent value in the feasible set, so as to minimize the number of inconsistent pairs across segment boundaries.

### Exhibit 1: 2D JIGSAW

1: Initialize the decimated set
2: Fix the field map value to be $\arg \min |p_i|$ for the pixel at the centroid of the foreground mask
3: Repeat till the decimated set includes all pixels
4: Run sum-product BP with soft priors from current decimated set
5: Enlarge the decimated set by including smooth segments of consistent field map values and with consistent boundary to the current decimated set
6: If inconsistent boundaries remain after running BP
7: Identify a smooth segment with inconsistent boundaries from the current decimated set
8: Swap the segment to minimize number of inconsistent pairs across segment boundary
9: Enlarge the decimated set by including the swapped segment

### 4.1. Message Passing

Sum-product belief propagation (BP) algorithm [6] is an iterative message-passing algorithm on a graphical model where messages are passed from a pixel to all its neighbors on the graph. For the field map estimation problem in this work, loopy BP [7] is used on a 2D image grid, where each pixel has four neighboring pixels. During an iteration, a message is passed from every pixel to its four neighbors, as illustrated...
in Fig. 1. Each message is a vector of dimension given by the number of feasible field map values at the receiving pixel, and each element of the message reflects the probability of the corresponding field map value.

![BP message passing on a four-connected 2D grid.](image)

For two neighboring pixels \( p \) and \( q \), \( \mathcal{N}(p) \) denotes the set of neighboring pixels to \( p \), and \( \mathcal{N}(p) \setminus q \) denotes the set of neighboring pixels to \( p \) other than \( q \). Let \( \mathcal{F}_p \) and \( \mathcal{F}_q \) denote the feasible sets of field map values at the pixels \( p \) and \( q \) respectively. Eq. 3 shows the computation of one element of the feasible sets of field map values at the pixels \( \psi \), particularly, it conveys the probability that the pixel \( q \) takes the value \( \psi_q \in \mathcal{F}_q \).

\[
m_{p \rightarrow q}^{t}(\psi_q) = \sum_{\psi_p \in \mathcal{F}_p} \left( V_{pq} \prod_{s \in \mathcal{N}(p) \setminus q} m_{s \rightarrow p}^{t-1}(\psi_p) \right) \tag{3}
\]

where

\[
V_{pq} = \exp\left(-\frac{(\psi_p - \psi_q)^2}{\sigma_q^2}\right), \tag{4}
\]

which denotes the interaction potential between two field map values \( \psi_p \) and \( \psi_q \). The interaction potential takes a Gaussian distribution and \( \sigma_q \) is the standard deviation of the distribution. Here the sum-product BP is implemented on 2D grid, which is the soft-decision counterpart to the recently proposed powerful energy minimization algorithm called max-product BP [8].

Each element of the message indicates the likelihood of the corresponding field map value, summed over the probability of each field map value for the sending pixel (hence the name sum-product). The interpretation of the message in Eq. 3 is as follows: the terms in the product computes the probability of node \( p \) taking field map value \( \psi_p \), based on beliefs from neighboring nodes other than \( q \). For each field map value \( \psi_p \), the probability of \( \psi_q \) at pixel \( q \) is computed via the interaction potential \( V_{pq} \). The overall probability of \( \psi_q \) is obtained by summing over the distribution of \( \psi_p \).

After \( T \) iterations (\( T \approx 30 \) in our implementation), a belief vector \( b_q \) is computed for the pixel \( q \). The element of \( b_q \) corresponding to the value \( \psi_q \) is given by

\[
b_q(\psi_q) = \prod_{p \in \mathcal{N}(q)} m_{p \rightarrow q}^{T}(\psi_q). \tag{5}
\]

Subsequently, for the pixel \( q \), the field map value \( \psi_q^* \) that corresponds to the largest element of the belief vector \( b_q \) is selected.

### 4.2. Decimation Move

Decimation refers to the fixing of field map values at one or more pixels. It is first proposed by statistical physicists to solve constraint satisfaction problems. By decimating successively, and running BP with prior beliefs from the decimated set, the algorithm converges to one specific solution of the problem. The first pixel, which is chosen at the centroid of the foreground mask, serves as the initial decimated set. The message from a decimated pixel \( p \) whose field map value is fixed to \( \psi_p \) to its neighboring free (not decimated) pixel \( q \) is

\[
m_{p \rightarrow q}^{t}(\psi_q) = V_{pq} \prod_{s \in \mathcal{N}(p) \setminus q} m_{s \rightarrow p}^{t-1}(\psi_p) \tag{5}
\]

for all \( t \). Only decimated pixels on the boundary of the decimated set sends messages. The messages serve as initial potential to guide the selection of field map values at the rest of the pixels. After the new field map values are chosen for the rest of the pixels based on the beliefs from the BP algorithm, the decimated set is enlarged to include pixels that satisfy the smoothness constraint in Eq. 2 with all their neighbors.

### 4.3. Swap Move

Due to the loopy nature of the BP algorithm, a large neighborhood of pixels can select field map values that are consistent among themselves, but ignores the guidance from the boundary values of the decimated set. This results in smooth segments bordering the decimated set, but retaining inconsistent boundaries between them. While a more carefully selected cost function might prevent the inconsistent boundaries, such a cost function is difficult to find in general, and is data-dependent. Instead, we propose the following “swap” move that exploits the characteristics of feasible field map values: the consistent solution can be obtained by shifting each field map value in the entire segment to its adjacent value in the feasible sets simultaneously.

Let \( S \) be a smooth segment within which the pixels are consistent among themselves; i.e., for any two neighboring pixels \( p \) and \( q \), \( |\psi_p - \psi_q| \) satisfies the smoothness constraint in Eq. 2. Let \( \psi_S \) denote the field map restricted to the segment \( S \). Suppose for each pixel the values in its feasible set is ordered from the smallest to the largest, and \( i_p \) denotes the index of the current value for \( \psi_p \) in the feasible set \( \mathcal{F}_p \). Define the operator \( \Gamma^k \) on the restricted field map \( \psi_S \) as the componentwise operator where

\[
\Gamma^k \psi_p = \mathcal{F}_p(i_p + k), \forall p \in S.
\]

where a positive \( k \) shifts each field map value in \( S \) to the \( k \)-th value after it in the feasible set, and a negative \( k \) shifts each field map value to the \( |k| \)-th value before it in the set. The
“swap” move locates the set of values under the operator $\Gamma^k$ that produces the smoothest boundary to the decimated set $D$:

$$\psi^*_S = \Gamma^j \psi_S$$

where

$$j = \arg\min_k \sum_{p \in D, q \in \mathcal{N}(p)} |\Gamma^j \psi_q - \psi_p|.$$ 

Hence, the shift results in $\psi^*_S$ (the new field map restricted to the segment $S$), which minimizes inconsistent pairs across the segment boundary.

5. RESULTS

To demonstrate the efficacy of the proposed field map estimation technique, we used a multi-echo GRE sequence to scan the ankle of a healthy volunteer at 3 T. Field map estimation of ankle scans at 3 T is challenging due to large field inhomogeneities from inadequate shim and/or susceptibility variations. A quadrature extremity coil was used with the following imaging parameters: $TE_{1,2,3} = 3.2, 6.4, 9.6$ ms, acquisition matrix $256 \times 256 \times 64$, FOV=20 x 20 x 5 cm, receiver bandwidth $\pm 83.3$ kHz, $TR = 11$ ms, and flip angle of 10°. Figure 2 shows the comparison of field maps (a and b) and separated water images (c and d) of a sample slice of this ankle study obtained using the proposed technique and IDEAL [1] implemented on the General Electric (GE) whole-body scanner. In comparison, the IDEAL implemented on the GE scanner produces a highly fragmented field map, while the proposed technique correctly resolves the field inhomogeneities in this challenging scenario and uniformly separates the diagnostically useful water image.

6. DISCUSSION AND CONCLUDING REMARKS

The proposed technique considers the periodic nature of feasible field map values and assembles a field map that spans multiple spectral FOVs. This is an advantage over the existing methods that implicitly assume sufficiently large spectral FOVs; hence, the proposed technique is particularly attractive for the several clinically important scenarios. The strength of the proposed technique lies in that no attempt is made to optimize a pre-determined cost function, given a lack of accurate models and cost functions. By separating the task of forming large, consistent segments and the task of assembling segments with minimum cost across boundaries, the proposed technique takes full advantage of the computational simplicity of the belief-propagation algorithm. As a result, the proposed technique can be naturally extended to 3D imaging, whereby the information from a completed 2D field map is successively propagated to produce 3D field map with little additional complexity.

Fig. 2. Comparison of field maps and separated water images of a sample slice of an ankle study at 3 T obtained using the proposed technique and on-line IDEAL field map estimation.

7. REFERENCES