1. Consider the following elliptic PDE

\[-\nabla \cdot (p(\vec{x}) \nabla u) = q(\vec{x}) \quad \text{for} \quad \vec{x} \in \Omega \subset \mathbb{R}^2,\]

\[u(\vec{x}) = 0 \quad \text{when} \quad \vec{x} \in \partial \Omega,\]

with \(p(\vec{x}) \geq m > 0\).

(a) Write the weak formulation of this problem in the form,

Find \(u \in V\), such that

\[a(u, v) = G(v), \quad \text{for all} \quad v \in V.\]

What is \(a(\cdot, \cdot)\)? What is \(G(\cdot)\)? What is \(V\)? Why are \(u\) and \(v\) in the same space? Is \(a(\cdot, \cdot)\) symmetric?

(b) Define the energy functional,

\[J(u) = \frac{1}{2} a(u, u) - G(u),\]

and show that the weak formulation above is equivalent to finding the “critical” or “stationary” point of \(J\), by computing

\[\frac{d}{d\tau} J(u + \tau v) \bigg|_{\tau = 0} = 0.\]

What is the second directional-derivative of \(J\)?

(c) Suppose the domain is a disjoint union of \(M\) triangles, \(E_1, E_2, \ldots, E_M\), with shared vertices. Further suppose that there are a total of \(N\) vertices with \(\vec{x} = (x, y)\) coordinates given by \(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N\). Show that the bilinear form, \(a(\cdot, \cdot)\), and linear functional, \(G(v)\), derived in the previous part can be written as a sum of integrals over each triangle (element).

(d) Consider approximating the solution to the weak formulation by restricting the solution space to a subspace spanned by \(N\) basis functions, \(\{\phi_1, \ldots, \phi_N\}\). Write the (approximate) solution as a linear combination of these basis functions,

\[u(\vec{x}) = \sum_{i=1}^{N} u_i \phi_i,\]

where \(\phi_i \in V\) and \(\text{span}(\phi_1, \ldots, \phi_N) \subset V\). Show that the minimizer of the energy, \(J(u)\), over the subspace \(\text{span}(\phi_1, \ldots, \phi_N)\), solves

\[\sum_{j=1}^{N} u_j a(\phi_j, \phi_i) = G(\phi_i), \quad \forall \ i = 1 \ldots N.\]

(e) Show that this approximation of \(u\) and \(v\) reduces the problem of solving the weak formulation to solving a linear system of equations of the form \(A\hat{u} = b\) where \(\hat{u}\) contains the unknown coefficients. What are the entries of \(A\) and \(b\) in terms of \(a(\cdot, \cdot)\) and \(G(\cdot)\)? Show that each entry of \(A\) and \(b\) can be written as a sum of integrals over triangles, if the domain is triangulated as described before.

(f) Suppose we modified the PDE to include nonhomogeneous Dirichlet boundary conditions,

\[-\nabla \cdot (p(\vec{x}) \nabla u) = q(\vec{x}) \quad \text{for} \quad \vec{x} \in \Omega \subset \mathbb{R}^2,\]

\[u(\vec{x}) = g(\vec{x}) \quad \text{when} \quad \vec{x} \in \partial \Omega,\]

with \(p(\vec{x}) \geq m > 0\). Show this yields the problem,

\[a(\hat{u}, v) = \hat{G}(v) \quad \text{for all} \quad v \in V,\]

with \(\hat{u} \in V\) and \(u := u^0 + \hat{u}\) and \(\hat{G}(v) := G(v) - a(u^0, v)\). Here, \(u^0\), is a known fixed function which satisfies the boundary conditions.
2. Here we derive the analytical formulae necessary for implementation of the finite element method using piecewise linear finite elements on a triangulated domain.

(a) Suppose \( \lambda_1 (\alpha, \beta) \) is the piecewise linear nodal basis function on the standard element (triangle), with \( \lambda_1 (0, 0) = 1 \) and \( \lambda_1 (1, 0) = \lambda_1 (0, 1) = 0 \). Show that when restricted to the standard triangle with vertices \((0, 0), (1, 0), (0, 1) \in \mathbb{R}^2\), this basis function can be written as

\[
\lambda_1 (\alpha, \beta) = 1 - \alpha - \beta
\]

and conclude that the gradient of \( \lambda_1 \) is

\[
\nabla \lambda_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.
\]

Derive a similar result for \( \lambda_2 \) which satisfies \( \lambda_2 (1, 0) = 1 \) and \( \lambda_2 (0, 0) = \lambda_2 (0, 1) = 0 \). Derive a similar result for \( \lambda_3 \) which satisfies \( \lambda_3 (0, 1) = 1 \) and \( \lambda_3 (0, 0) = \lambda_3 (1, 0) = 0 \).

(b) Given three vertices of a general triangle, \( \vec{x}_1, \vec{x}_2, \vec{x}_3 \in \mathbb{R}^2 \), write down an affine transformation, \( T \), that maps the vertices of the “standard triangle” \((0, 0), (1, 0), (0, 1) \in \mathbb{R}^2\) to the “general triangle”. Show that the Jacobian (derivative) of this transformation is

\[
J_T = \begin{bmatrix} (\vec{x}_2 - \vec{x}_1) & (\vec{x}_3 - \vec{x}_1) \end{bmatrix} \in \mathbb{R}^{2 \times 2},
\]

where \( \vec{x} = (x, y) \).

(c) Suppose \( \phi_r \) is the piecewise linear nodal basis function for a general triangulation with \( \phi_r (\vec{x}_r) = 1 \) and \( \phi_r (\vec{x}_s) = 0 \) if \( r \neq s \). Using the results of the prior two parts show that when restricted to the “general triangle” that \( \phi_r \) can be written as a composition of \( \lambda_r \) and \( T^{-1} \),

\[
\phi_r (\vec{x}) = \lambda_r (T^{-1} (\vec{x}))
\]

and the gradient of \( \phi_r \) is

\[
\nabla \phi_r |_{\vec{x}} = J_T^{-T} \nabla \lambda_r
\]

where \( \nabla \lambda_r \) is constant.

(d) Let \( \vec{x} = (x, y) \in \mathbb{R}^2 \), \( \vec{\alpha} = (\alpha, \beta) \in \mathbb{R}^2 \), \( E \) the region defined by the general triangle, \( S \) the region defined by the standard triangle, \( T \) the affine transformation defined above, and \( |J_T| \) the determinant of the Jacobian of the transformation.

Show that

\[
\langle p (\vec{x}) \nabla \phi_s, \nabla \phi_r \rangle_E = (J_T^{-T} \nabla \lambda_r)^T (J_T^{-T} \nabla \lambda_s) \ |J_T| \int_S p (T (\vec{\alpha})) \ d\vec{\alpha}
\]

and

\[
\langle q (\vec{x}), \phi_r \rangle_E = |J_T| \int_S q (T (\vec{\alpha})) \lambda_r (\vec{\alpha}) \ d\vec{\alpha}
\]

using the change of variables formula

\[
\int_E h (\vec{x}) \ d\vec{x} = \int_S h (T (\vec{\alpha})) \ |J_T| \ d\vec{\alpha}
\]