A space efficient streaming algorithm for triangle counting using the birthday paradox

[Extended Abstract]

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ABSTRACT
We design a space efficient algorithm that approximates the transitivity (global clustering coefficient) and total triangle count with only a single pass through a graph given as a stream of edges. Our procedure is based on the classic probabilistic result, the birthday paradox. When the transitivity is constant and there are more edges than wedges (common properties for social networks), we can prove that our algorithm requires $O(\sqrt{n})$ space ($n$ is the number of vertices) to provide accurate estimates. We run a detailed set of experiments on a variety of real graphs and demonstrate that the memory requirement of the algorithm is a tiny fraction of the graph. For example, even for a graph with 200 million edges, our algorithm stores just 60,000 edges to give accurate results. Being a single pass streaming algorithm, our procedure also maintains a real-time estimate of the transitivity/number of triangles of a graph, by storing a miniscule fraction of edges.

1. INTRODUCTION
Triangles are one of the most important motifs in real world networks. Whether the networks come from social interaction, computer communications, financial transactions, proteins, or ecology, the abundance of triangles is pervasive and it is critical feature that distinguishes real graphs from random graphs. There is a rich body of literature on analysis of triangles and counting algorithms. Social scientists use triangle counts to understand graphs [16, 26, 10, 37]; graph mining applications such as spam detection and finding common topics on the WWW use triangle counts [17, 6]; motif detection in bioinformatics often count the frequency of triadic patterns [23]. Nevertheless, counting triangles continues to be a challenge due to sheer size of the graphs (easily in the order of billions of edges).

Many massive graphs come from modeling interactions in a dynamic system. People call each other on the phone, exchange emails, or become part of a tight unit (e.g, co-authoring a paper); computers exchange messages; animals come in the vicinity of each other; companies trade with each other. These interactions manifest as a stream of edges. The edges appear with timestamps, or “one at a time.” The network (graph) that represents the system is an accumulation of the observed edges. There are many methods to deal with such massive graphs, such as random sampling [27, 34, 29], MapReduce paradigm [30, 25], distributed-memory parallelism [3, 11], adopting external memory [12, 2], and multithreaded parallelism [8].

But all of these methods need to store at least a large fraction of the data. A small space streaming algorithm maintains a very small (using some randomness) set of edges, called the “sketch” at any given time, and updates this sample as edges appear. Based on the sketch and some auxiliary data structures, the algorithm computes an accurate estimate for the number of triangles for the graph seen so far. The sketch size is orders of magnitude smaller than the total graph. Furthermore, it can be updated rapidly when new edges arrive, and hence maintains a real-time estimate of the number of triangles. We also want a single pass algorithm, so it only observes each edge once (think of it as making a single scan of a log file). The algorithm cannot
1.2 Results

respectively as transitivity of guarantee of maintaining a running estimate of triangles and algorithm keeps a small “sketch” of the edges it has seen. The passed by. The memory algorithm is never allowed to see an edge that has already can choose to update data structures in the edges in stream order. At each edge A streaming algorithm has a small memory triangles \[ T \] of vertex labels. So new vertices are implicitly added as new labels. There is no assumption on the order of edges (This is also called the global clustering coefficient \[ 29 \].) Formal, a single pass streaming algorithm is defined as follows. Consider a sequence of (distinct) edges \( e_1, e_2, \ldots, e_m \). Let \( G_t \) be the graph at time \( t \), formed by the edge set \( \{ e_i \mid i \leq t \} \). The stream of edges can be thought of as a sequence of edge insertions into the graph. (Vertex insertions can be trivially handled.) We do not know the number of vertices ahead of time, and simply see each edge as a pair \( (u, v) \) of vertex labels. So new vertices are implicitly added as new labels. There is no assumption on the order of edges in the stream. Edges incident to a single vertex do not necessarily appear together.

In this paper, we do not consider edge/vertex deletions or repeated edges. In that sense, this is a simplified version of the full-blown streaming model. Nonetheless, the edge insertion model is what is studied in previous work on counting triangles \( [5, 19, 9, 1, 20] \).

A streaming algorithm has a small memory \( M \), and sees the edges in stream order. At each edge \( e_t \), the algorithm can choose to update data structures in \( M \) (using the edge \( e_t \)). Then the algorithm proceeds to \( e_{t+1} \), and so on. The algorithm is never allowed to see an edge that has already passed by. The memory \( M \) is much smaller than \( m \), so the algorithm keeps a small “sketch” of the edges it has seen. The aim is to estimate the number of triangles in \( G := G_m \) at the end of the stream. Usually, we desire the more stringent guarantee of maintaining a running estimate of triangles and transitivity of \( G_t \) at time \( t \). We denote these quantities respectively as \( T_t \) and \( \kappa_t \).

1.1 The streaming setting

Let \( G \) be a simple undirected graph with \( n \) vertices and \( m \) edges. Let \( T \) denote the number of triangles in the graph and \( W \) be the number of wedges, where a wedge is a path of length 2. A common measure is the transitivity \( \kappa = 3T/W \) \[ 36 \], a measure of how often friends of friends are also friends. (This is also called the global clustering coefficient \( 29 \).) We prove a space bound of \( O(\sqrt{m/T}) \)-space algorithm to provably estimate the transitivity (with arbitrary additive error) in a streaming graph. Streaming algorithms for counting triangles or computing the transitivity have been studied before, but no previous algorithm attains this space guarantee. (Buriol et al \[ 9 \] give a single pass algorithm with a stronger relative error guarantee that requires space \( O(mn/T) \). We discuss in more detail later.)

Although our theoretical result is interesting asymptotically, the constant factors and dependence on error in our bound are large. Our main result is a practical streaming algorithm (based on the theoretical one) for computing \( \kappa \) and \( T \), using some additional probabilistic heuristics. We perform an extensive empirical analysis of our algorithm on a variety of datasets (many thanks to SNAP \[ 39 \], for their extensive collection of downloadable graphs). The salient features of our algorithm are:

- Theoretical basis: Our algorithm is based on the classic birthday paradox: if we choose 23 random people, the probability that 2 of them share a birthday is at least 1/2 (Chap. II.3 of \[ 18 \]). We extend this analysis for sampling wedges in a large pool of edges. The final streaming algorithm is designed by using reservoir sampling with wedge sampling \[ 29 \] for estimating \( \kappa \). We prove a space bound of \( O(m/\sqrt{T}) \), which we show is \( O(\sqrt{n}) \) under common conditions for social networks.

While our theory appears to be a good guide in designing the algorithm and explaining its behavior, it should not be used to actually decide space bounds in practice. Also, we point out that for graphs where \( T \) is small, our algorithm does not provide good guarantees for small space (since \( m/\sqrt{T} \) is large).

- Accuracy and scalability with small sketches: We test our algorithm on a variety of graphs from different sources. In all instances, we get accurate estimates for \( \kappa \) and \( T \) by storing at most 40K edges. This is even for graphs where \( m \) is in the order of millions. Our relative errors on \( \kappa \) and the number of triangles are mostly less than 5%. (In a graph with very few triangles where \( \kappa < 0.01 \), our triangle

Figure 1: Realtime tracking of number of triangles and transitivities on cit-Patents (16M edges), storing only 60K edges from the past. (Absolute values are shown with time (in years) on the x-axis.)

We present a single pass, \( O(m/\sqrt{T}) \)-space algorithm to provably estimate the transitivity (with arbitrary additive error) in a streaming graph. Streaming algorithms for counting triangles or computing the transitivity have been studied before, but no previous algorithm attains this space guarantee. (Buriol et al \[ 9 \] give a single pass algorithm with a stronger relative error guarantee that requires space \( O(mn/T) \). We discuss in more detail later.)
estimate has relative error of 12%. Our algorithm processes extremely large graphs. Our experiments include a run on a streamed Orkut social network with 200M edges (by storing only 40K edges, relative errors are at most 4%). We get similar results on streamed Flickr and Live-journal graphs with tens of millions of edges.

We run detailed experiments on some test graphs (with 1-3 million edges) with varying parameters to show convergence of our algorithm. Comparisons with previous work [9] show that our algorithm gets within 5% of the true answer, while the previous algorithm is off by more than 50%.

• Real-time tracking: For a temporal graph, our algorithm precisely tracks both κ and T with less storage. By storing 60K edges of the past, we can track this information for a patent citation network with 16 million edges [39]. Refer to Fig. 1. We maintain a real-time estimate of both the transitivity and number of triangles with a single pass, storing less than 1% of the graph. We see some fluctuations in the transitivity estimate due to the randomness of the algorithm, but the overall tracking is fairly accurate.

2. PREVIOUS WORK

Enumeration of all triangles is a well-studied problem [13, 28, 22, 7, 14]. Recent work by Cohen [15], Suri and Vassilvitskii [30], Arifuzzaman et al. [3] give massively parallel implementations of these algorithms. Eigenvalue/trace based methods have also been used [31, 4] to compute estimates of the total and per-degree number of triangles.

Tsourakakis et al. [32] started the use of sparsification methods, the most important of which is Doullon [34]. Various analyses of this algorithm (and its variants) have been proposed [21, 33, 38, 24]. Wedge-sampling based algorithms provide provable accurate estimations on various triadic measures on graphs [27, 29].

Theoretical streaming algorithms for counting triangles were initiated by Bar-Yossef et al. [5]. Subsequent improvements were given in [19, 9, 1, 20]. The space bounds achieved are of the form mn/T. Note that m/√T ≤ mn/T whenever T ≤ n^2 (which is a reasonable assumption for sparse graphs). These algorithm are rarely practical, since T is often much smaller than mn. (Some multi-pass streaming algorithms give stronger guarantees, but we will not discuss them here.)

Buriol et al. [9] give an implementation of their algorithm. For almost all of their experiments on graphs, with storage of 100K edges, they get fairly large errors (always more than 10%, and often more than 50%). Buriol et al provide an implementation in the incidence list setting, where all neighbors of a vertex arrive together. In this case, their algorithm is quite practical since the errors are quite small. Our algorithm scales to sizes (100 million edges) larger than their experiments. We get better accuracy with far less storage, without any assumption on the ordering of the data stream. Furthermore, our algorithm performs accurate real-time tracking.

Becchetti et al. [6] gave a semi-streaming algorithm for counting the triangles incident to every vertex. Their algorithm uses clever methods to approximate Jaccard similarities, and requires multiple passes over the data.

3. OUTLINE

We begin in §4 by providing an intuitive high-level explanation of the algorithm. §5 gives a formal description of our implemented algorithm (denoted STREAMING-Triangles). The theoretical analysis is done in §6, for an idealized (and inefficient) variant called Single-Bit. We stress that Single-Bit is a thought experiment to highlight the theoretical aspects of our result, and we do not actually implement it. In §6.1, we explain the heuristics used to get STREAMING-Triangles. The remainder of §6 gives a mathematical analysis of Single-Bit.

Finally, in §7, we give results on our runs of STREAMING-Triangles on real graphs.

4. INTUITION FOR THE ALGORITHM

The starting point is the idea of wedge sampling to estimate κ [29]. A wedge is closed if it participates in a triangle and open otherwise. Note that κ = 3T/W is exactly the probability that a uniform random wedge is closed. This gives a simple randomized algorithm for estimating κ (and T), by generating a set of (independent) uniform random wedges and finding the fraction that are closed. But how do we sample wedges from a stream of edges?

Suppose we just sampled a uniform random set of edges. How large does this set need to be to get a wedge? The birthday paradox can be used to deduce that (as long as W ≥ m, pretty much always true for real networks) O(√m) edges suffice. A more sophisticated result, given in Lem. 1, provides (weak) concentration bounds on the number of wedges generated by a random set of edges. A “small” number of uniform random edges can give enough wedges to perform wedge sampling (which in turn is used to estimate κ).

A set of uniform random edges can be maintained by the standard method of reservoir sampling [36]. From these edges, we generate a random wedge by doing a second level of reservoir sampling. This implicitly treats the wedges created in the edge reservoir as a stream, and performs reservoir sampling on that. Overall, this approximates uniform random wedge sampling.

As we maintain our reservoir wedges, we check for closure by the future edges in the stream. But there are closed wedges that cannot be verified, because the closing edge may have already appeared in the past. A simple observation used by past streaming algorithms saves the day [19, 9]. In each triangle, there is exactly one wedge whose closing edge appears in the future. So we try to approximate the fraction of these “future closed” wedges, which is exactly one-third of the fraction of closed wedges. (Hence, the factor 3 that pops up in STREAMING-Triangles.)

Finally, to estimate T from κ, we need an estimate on the total number of wedges W. This can be obtained by reverse engineering the birthday paradox: given the number of wedges in our reservoir sample of edges, we can estimate W (again, using the workhorse Lem. 1).
5. THE PROCEDURE STREAMING-TRIANGLES

The streaming algorithm maintains two primary sets of data: the edge reservoir and the wedge reservoir. These are sets of edges and wedges that the algorithm stores from the past. The parameters for the streaming algorithm are the respective sizes of these sets, denoted by \( s_e \) and \( s_w \) respectively. The main algorithm is described in STREAMING-TRIANGLES, although most of the technical computation is performed in Update (which is invoked every time a new edge appears). After processing edge \( e_t \), the algorithm computes running estimates for \( \kappa_t \) and \( T_t \). These values do not have to be stored, so they are immediately output and forgotten. We describe the main data structures of the algorithm STREAMING-TRIANGLES.

- Array \( edge\_res[1 \ldots s_e] \): This is the array of reservoir edges and is the primary subsample of the stream maintained.
- New wedges \( N_t \): This is a list of all wedges involving \( e_t \) formed only by edges in \( edge\_res \). This may often be empty, if \( e_t \) is not in \( edge\_res \). We do not necessarily maintain this list explicitly, and we discuss implementational details later.
- Variable \( tot\_wedges \): This is the total number of wedges formed by edges in \( edge\_res \).
- Array \( wedge\_res[1 \ldots s_w] \): This is an array of reservoir wedges of size \( s_w \).
- Array \( isClosed[1 \ldots s_w] \): This is a boolean array. We set \( isClosed[i] \) to be true if wedge \( wedge\_res[i] \) is found to be closed.

Algorithm 1: STREAMING-TRIANGLES(\( s_e, s_w \))
1. Initialize \( edge\_res \) of size \( s_e \) and \( wedge\_res \) of size \( s_w \). For each edge \( e_t \) in stream,
2. Call Update(\( e_t \)).
3. Let \( \rho \) be the fraction of entries in \( isClosed \) set to true.
4. Set \( \kappa_t = 3\rho \).
5. Set \( T_t = \lceil \rho t^2 / s_e (s_e - 1) \rceil \times \) tot\_wedges.

Algorithm 2: Update(\( e_t \))
1. For every reservoir wedge \( wedge\_res[i] \) closed by \( e_t \), set \( isClosed[i] = \text{true} \).
2. Flip a coin with heads probability \( = 1 - (1 - 1/t)^{x_t} \). If it flips to tails, stop and proceed to the next edge in stream. (So remaining code is processed only if coin comes heads.)
3. Choose a uniform index \( i \in [s_e] \). Set \( edge\_res[i] = e_t \).
4. Determine \( N_t \) and let \( new\_wedges = |N_t| \).
5. Update tot\_wedges, the number of wedges formed by \( edge\_res \).
6. Set \( q = new\_wedges / tot\_wedges \).
7. For each index \( i \in [s_w] \),
8. Flip coin with heads probability \( q \).
9. If tails, continue to next index in loop.
10. Pick uniform random \( w \in N_t \) that involves \( e_t \).
11. Replace \( wedge\_res[i] = w \). Reset \( isClosed[i] = \text{false} \).

5.1 Implementation details

Computing \( \kappa_t \) and \( T_t \) are simple and require no overhead. We maintain \( edge\_res \) as a time-variable subgraph. Each time \( edge\_res \) is updated, the subgraph undergoes an edge insert and edge delete. Suppose \( e_t = (u, v) \). Wedges in \( N_t \) are given by the neighbors of \( u \) and \( v \) in this subgraph. From random access to the neighbor lists of \( u \) and \( v \), we can generate a random wedge from \( N_t \) efficiently.

Updates to \( edge\_res \) are very infrequent. At time \( t \), the probability of an update is \( 1 - (1 - 1/t)^{x_t} \). By linearity of expectation, the total number of times that \( N_t \) is non-empty is
\[
\sum_{t \leq m} 1 - (1 - 1/t)^{x_t} \approx \sum_{t \leq m} s_e / t \approx s_e \ln m
\]
For a fixed \( s_e \), this increases very slowly with \( m \). So for most steps, we neither update \( edge\_res \) or sample a new wedge.

The total number of edges that are stored from the past is \( s_e + s_w \). The edge reservoir explicitly stores edges, and at most \( s_w \) edges are implicitly stored (for closure). Regardless of the implementation, the extra data structures overhead is at most twice the storage parameters \( s_e \) and \( s_w \). Since these are at least 2 orders of magnitude smaller than the graph, this overhead is easy to pay for.

6. THE IDEALIZED ALGORITHM SINGLE-BIT

The algorithm called SINGLE-BIT is an idealized variant of STREAMING-TRIANGLES that we can formally prove theorems about. It requires more memory and expensive updates, but explains the basic principles behind our algorithm. We later give the memory reducing heuristics that take us from SINGLE-BIT to STREAMING-TRIANGLES.

The procedure SINGLE-BIT has a single space parameter \( s \) and outputs a single (random) bit at each \( t \). The expectation of this bit is related to the transitivity \( \kappa_t \). We will describe SINGLE-BIT in a more mathematical fashion. SINGLE-BIT maintains a set of edges \( R_t \) of size \( s \). The set of wedges constructed from \( R_t \) is \( W_t \). Formally, \( W_t = \{ \text{wedge } (e, e') | e, e' \in R_t \} \). For every \( e_t \), SINGLE-BIT records all wedges closed by this edge. SINGLE-BIT maintains a set \( C_t \), the set of wedges in \( W_t \) for which it has detected a closing edge. (Note that this is a subset of all closed wedges in \( W_t \).) This set is easy to update as \( R_t \) changes.

Algorithm 3: SINGLE-BIT(\( s \))
1. Initialize \( R_0 \) as a set of \( s \) dummy edges.
2. For each \( e_t \) in stream,
3. For each edge in \( R_{t-1} \), replace it (independently) by \( e_t \) with probability \( 1/t \). This yields \( R_t \).
4. Construct the set of wedges \( W_t \).
5. Let \( C_t \) be the set of all wedges in \( W_t \) closed by \( e_t \).
6. Add all wedges in \( (C_{t-1} \cap W_t) \) to \( C_t \). If \( W_t \) is empty, output \( b_t = 0 \) and continue to next edge.
7. Pick a uniform random wedge in \( W_t \).
8. Output \( b_t = 1 \) if this wedge is in \( C_t \) and \( b_t = 0 \) otherwise.

For convenience, we state this theorem for the final time step. However, it also holds (with an identical proof) for any large enough time \( t \).
Theorem 1. Assume $W \geq m$. Suppose $s \geq cm/(\beta^3 \sqrt{T})$, for some sufficiently large constant $c$. Set $\varepsilon = m^2 |W_m| / (s(s-1))$. Then $|\bar{k}/3 - E[b_m]| < \beta$ and with probability $> 1 - \beta$, $|W - \varepsilon s| < \beta W$.

We now show that $m/\sqrt{T} = O(\sqrt{n/\kappa})$ (usually much smaller for heavy tailed graphs) when $W \geq m$. Denote the degree of vertex $v$ by $d_v$. In this case, we can bound $2W = \sum d_v (d_v - 1) = \sum d_v^2/2 - \sum d_v^2 / 2W$, so $W \geq \sum d_v^2 / 4$. By $2m = \sum d_v$, the Cauchy-Schwarz inequality,

$$m/\sqrt{W} \leq \sqrt{\sum d_v^2 / \sum d_v^2} \leq \sqrt{\sqrt{\sum d_v^2} / \sqrt{\sum d_v^2}} = \sqrt{n}$$

Using the above bound, we get $m/\sqrt{T} = \sqrt{3m/\sqrt{kW}} \leq \sqrt{3n/\kappa}$. Hence, when $W \geq m$ and $\kappa$ is a constant (both reasonable assumptions for social networks), we require only $O(\sqrt{n})$ space.

6.1 Circumventing problems with SINGLE-Bit

SINGLE-Bit has two main problems, which are handled in STREAMING-Triangles by heuristic arguments.

Thm. 1 immediately gives a small sublinear space streaming algorithm for estimating $\kappa$. The output of SINGLE-Bit has the right expectation. We can run many independent invocations of SINGLE-Bit and take the fraction of 1s output to estimate $E[b_m]$ (which in turn is close to $\bar{k}/3$). A Chernoff bound tells us that $O(1/\varepsilon^2)$ invocations suffice to estimate $E[b_m]$ within additive error $\varepsilon$. The total space becomes $O(m/(\sqrt{T} \varepsilon^2))$, which can be very expensive in practice. Even though $m/\sqrt{T}$ is not large, for the reasonable value of $\varepsilon = 0.01$, the storage cost blows up by a factor of $10^4$. This is the standard method used in previous work for streaming triangle counts.

This blowup is avoided in STREAMING-Triangles by reusing the same reservoir of edges for sampling wedges. Note that SINGLE-Bit is trying to generate a single uniform random wedge from $G$, and we use independent reservoirs of edges to generate multiple samples. Lem. 1 says that for a reservoir of $km/\sqrt{W}$ edges, we expect $k^2$ wedges. So, if $k > 1/\varepsilon$ and we get $> 1/\varepsilon^2$ wedges. Since the reservoir contains a large set of wedges, we could just use a subset of these for estimating $E[b_m]$. Unfortunately, these wedges are correlated with each other, and we cannot theoretically prove the desired concentration. In practice, it appears that these wedges are sufficiently uncorrelated that we get excellent results by reusing the reservoir. This is an important distinction from past streaming work [19, 9]. We can multiply our space by $1/\varepsilon$ to (heuristically) get error $\varepsilon$, but this is not possible through previous algorithms. Their space is multiplied by $1/\varepsilon^2$.

The second issue is that SINGLE-Bit requires a fair bit of bookkeeping. We need to generate a random wedge from the large set $W_t$. While this is possible by storing edge$_{\text{res}}$ as a subgraph, we have a nice (at least in the authors’ opinion) heuristic fix that avoids these complications.

Suppose we have a uniform random wedge $w \in W_{t-1}$. We can convert it to an “almost” uniform random wedge in $W_t$. If $W_t = W_{t-1}$ is true (so $N_t = \emptyset$ which is true most of the time), then $w$ is also uniform in $W_t$. Suppose not. Note that $W_t$ is constructed by removing some wedges from $W_{t-1}$ and inserting $N_t$. Since $w$ is uniform random in $W_{t-1}$, if $w$ is also present in $W_t$, then it is uniform random in $W_t \setminus N_t$. Replacing $w$ by a uniform random wedge in $N_t$ with probability $|N_t|/|W_t|$ yields a uniform random wedge in $W_t$. This is precisely what STREAMING-Triangles does.

When $w \notin W_t$, then the edge replaced by $e_t$ must be in $w$. We approximate this as a low probability event and simply ignore this case. Hence, in STREAMING-Triangles, we simply assume that $w$ is always in $W_t$. This is technically incorrect, it appears to have little effect on the accuracy in practice. And it leads to a cleaner, efficient implementation.

6.2 Proving Thm. 1

We begin with some preliminaries. First, the set $R_t$ is a set of $s$ uniform i.i.d. samples from $\{e_1, e_2, \ldots, e_t\}$, a direct consequence of reservoir sampling. Next, we define future-closed wedges. Take the final graph $G$ and label all edges with their timestamp. For each triangle, the wedge formed by the earliest two timestamps is a future-closed wedge. In other words, if a triangle $T$ has edges $e_i, e_j, e_k$ ($i < j < k$), then the wedge $\{e_i, e_j\}$ is future-closed. The number of future-closed wedges is exactly $T$, since each triangle contains a single such wedge. We have a simple yet important claim about SINGLE-Bit.

Claim 1. The set $C_m$ is exactly the set of future closed wedges in $W_m$.

Proof. Consider some wedge $\{e_i, e_j\}, i < j$ in $W_m$. This wedge was formed at time $j$, and remains in all $W_t$ for $j \leq t \leq m$. If this wedge is future closed (say by edge $e_{t'}$, for $t' > j$), then at time $t'$, the wedge will be detected to be closed. Since this information is maintained by SINGLE-Bit, the wedge will be in $C_m$. If the wedge is not future closed, then no closing edge will be found for it after time $j$. Hence, it will not be in $C_m$. □

The main technical effort goes into showing that the number of wedges formed by edges in $R_m$ (this is precisely $|W_m|$) can be used to determine the actual number of wedges in $G_m$. Furthermore, the number of future closed wedges in $R_m$ (precisely $|C_m|$), by Claim 1 can be used to estimate $T$.

This is formally expressed in the next lemma. Roughly, if $s = km/\sqrt{W}$, then we expect $k^2$ wedges formed by $R_m$. We also get weak concentration bounds for the quantity. A similar bound (with somewhat weaker concentration) holds even when we consider the set of future closed wedges.

Lemma 1 (Birthday paradox for wedges). Let $G$ be a graph with $m$ edges and $S$ be some fixed subset of edges in $G$. Let $R$ be a set of $s$ uniformly and independently selected edges (with replacement) from $G$. Let $X$ be the random variable denoting the number of wedges in $S$ formed by edges in $R$.

1. $E[X] = \binom{s}{2} |S| / m^2$.
2. Let \( \beta > 0 \) be a parameter and \( c' \) be a sufficiently large constant. Assume \( W \geq m \). If \( s \geq c'm/(\beta^3\sqrt{W}) \), then with probability at least \( 1 - \beta \), \( |X - E[X]| \leq \frac{(\beta W/|S|)E[X]}{h^2} \).

**Proof.** The first part is an adaptation of the birthday paradox calculation. Let the set \( R = \{ r_1, r_2, \ldots, r_s \} \). We define random variables \( X_{i,j} \) for each \( i, j \in [s] \) with \( i < j \). Let \( X_{i,j} = 1 \) if the wedge \( \{ r_i, r_j \} \) belongs to \( S \) and 0 otherwise. Then \( X = \sum_{i<j} X_{i,j} \).

Since \( R \) consists of uniform i.i.d. edges from \( G \), the following holds: for every \( i < j \) and every (unordered) pair of edges \( \{ e_a, e_b \} \) from \( E \), \( \Pr[X_{i,j}] = \Pr[\{ e_a, e_b \}] = 2/m^2 \). This implies \( \Pr[X_{i,j} = 1] = 2|S|/m^2 \). By linearity of expectation, we have \( E[X] = \binom{s}{2}E[X_{1,2}] = \binom{s}{2}\Pr[X_{1,2} = 1] = \binom{s}{2}(2|S|/m^2) \), as required.

The second part is obtained by applying the Chebyshev inequality. Let \( \Var[X] \) denote the variance of \( X \). For any \( h > 0 \),

\[
\Pr[|X - E[X]| > h] \leq \Var[X]/h^2 \tag{1}
\]

We need an upper bound on the variance of \( X \) to apply (1). The proof of the following bound on \( \Var[X] \) is fairly technical and deferred to §6.3.

**Lemma 2 (Variance bound).** Assuming \( W \geq m \) and \( s \geq m/\sqrt{W} \), \( \Var[X] \leq c's^3W^{3/2}/m^3 \) (for sufficiently large constant \( c' \)).

We set \( h = (\beta W/|S|)E[X] \). Note that \( E[X] = s(s-1)|S|/m^2 \). By (1), \( \Pr[|X - E[X]| > h] \) is at most the following:

\[
\frac{\Var[X]}{h^2} \leq \frac{c's^3W^{3/2}/m^3}{(\beta^2W^2E[X])^2/|S|^2} = \frac{c's^3|S|^2}{\beta^2\sqrt{W}m^3(s(s-1)|S|/m^2)^2} \leq \frac{c'm/(\beta^3\sqrt{W})}{s} \leq \beta
\]

where the final inequality holds by the bound on \( s \). \( \Box \)

We sketch the proof of Thm. 1. (Details in §6.4.) At the end of the stream, the output bit \( b_m \) is 1 if \( |W_m| > 0 \) and a wedge from \( C_m \) is sampled. Note that both \( |W_m| \) and \( |C_m| \) are random variables.

To deal with the first event, we apply Lem. 1 with \( S \) being the set of all wedges. So, \( E[|W_m|] = \binom{s}{2}(2W/m^2) \). Since \( s \geq c'm/(\beta^3\sqrt{W}) \geq c'm/(\beta^3\sqrt{W}) \), \( E[|W_m|] \geq c'^2/\beta^6 \) (a fairly large number). Intuitively, the probability that \( |W_m| = 0 \) should be very small, and this can be bounded using the concentration bound of Lem. 1.

For this informal discussion, let us just assume that \( |W_m| > 0 \) happens. Now, the probability that \( b_m = 1 \) (which is \( E[b_m] \)) is exactly the fraction \( |C_m|/|W_m| \). Suppose both these behave like their expectation. By Claim 1, \( C_m \) is the set of future closed wedges. The number of future closed wedges is \( T \), so Lem. 1 tells us that \( E[|C_m|] = \binom{s}{2}(2T/m^2) \). Hence, \( E[|C_m|]/E[|W_m|] = T/W = \kappa/3 \).

In general, the value of \( |C_m|/|W_m| \) might be quite different from \( E[|C_m|]/E[|W_m|] \). Note that \( E[|C_m|] \geq c'/\beta^6 \), by choice of \( s \). Using the concentration bounds of Lem. 1, we can argue that the deviation of \( |C_m|/|W_m| \) from \( E[|C_m|]/E[|W_m|] \) is at most \( \beta \) with probability \( 1 - \beta \).

### 6.3 Bounding the variance: proof of Lem. 2

**Proof.** We use the same notation as in the proof of Lem. 1. For convenience, we set \( \mu = E[X_{i,j}] \), which is \( 2|S|/m^2 \). By the definition of variance and linearity of expectation,

\[
\Var[X] = E[X^2] - (E[X])^2
\]

By (1),

\[
\Var[X] = \sum_{i<j} X_{i,j} \sum_{p<q} X_{p,q} - \binom{s}{2} \mu^2 = \sum_{i<j, p<q} E[X_{i,j}X_{p,q}] - \binom{s}{2} \mu^2.
\]

The summation is split as follows.

\[
\sum_{i<j, p<q} E[X_{i,j}X_{p,q}]
\]

We deal with each of these terms separately. For convenience, we refer to the terms (in order) as \( A_1, A_2, A_3 \). We first list the upper bounds for each of these terms and derive the final bound on \( \Var[X] \).

- \( A_1 = \binom{s}{2} \mu \).
- \( A_2 \leq 12 \binom{s}{2} \sum_{v<m} d_v^3/m^3 \).
- \( A_3 = 6 \binom{s}{2} \mu^2 \).

We shall prove these shortly. From these, we directly bound \( \Var[X] \).

\[
\Var[X] = A_1 + A_2 + A_3 \leq \left( \frac{s}{2} \right)^2 \mu^2 \leq \left( \frac{s}{2} \right)^2 \mu^2 + 12 \binom{s}{2} \frac{d_v^3}{m^3} + 6 \binom{s}{2} \mu^2 - \left( \frac{s}{2} \right)^2 \mu^2
\]

Note that \( 6 \binom{s}{2} = s(s-1)(s-2)(s-3)/4 \leq \lceil s(s-1)/2 \rceil \).

Since \( W \geq m \), we have \( \sum_v d_v^3 \leq 2 \sum_v \binom{s}{2} + 2m \leq 4W \).

Plugging these bounds in (and using gross upper bounds to ignore constants),

\[
\Var[X] \leq s^2 \mu + 2s^3 \frac{\sum_v d_v^3}{m^3} \leq 2s^2 |S| + 16s^3 W^{3/2}/m^3 \leq 2s^2 W/m^2 + 16s^3 W^{3/2}/m^3 \leq 18s^3 W^{3/2}/m^3
\]

The final step uses the fact that \( s \geq m/\sqrt{W} \). \( \Box \)
We now bound the terms $A_1$, $A_2$, and $A_3$ in three separate claims.

**Claim 2.** $\sum_{i<j} E[X_{i,j}^2] = \binom{s}{2} \mu$.

**Proof.** Since $X_{i,j}$ only takes the values 0 and 1, $E[X_{i,j}^2] = E[X_{i,j}] = \mu$. \qed

**Claim 3.**

$$\sum_{(i,j) \in \mathcal{S}} E[X_{i,j}X_{p,q}] \leq 12 \binom{s}{3} \sum_{e \in [n]} d_e^3 / m^3.$$  

**Proof.** How many terms are in the summation? There are 3 distinct indices in $(i,j,p,q)$. For each distinct triple of indices, there are 6 possible way of choosing $i < j$, $p < q$ among these indices such that $(i,j) \cap (p,q) = 1$. This gives 6 terms. By symmetry each term in the summation is equal to $E[X_{i,j}X_{i,j}]$. This is exactly the probability that $(r_1,r_2) \in \mathcal{S}$ and $(r_1,r_3) \in \mathcal{S}$. We bound this probability above by $2 \sum_{e \in [n]} d_e^3 / m^3$, completing the proof. \qed

Let $\mathcal{E}$ be the event $\{r_1,r_2\} \in \mathcal{S}$ and $\{r_1,r_3\} \in \mathcal{S}$. Let $\mathcal{F}$ be the event that edge $r_1$ intersects edges $r_2$ and $r_3$. Observe that $\mathcal{E}$ implies $\mathcal{F}$. Therefore, it suffices to bound the probability of the latter event. We also use the inequality $\forall a, b, (a+b)^2 \leq 2(a^2 + b^2)$.

$$\Pr[\{r_1,r_2\} \in \mathcal{S}, \{r_1,r_3\} \in \mathcal{S}] \leq \Pr[\{r_1,r_2\} \notin \mathcal{S}, \{r_1,r_3\} \notin \mathcal{S}]$$

$$= \sum_{(u,v) \in E} \Pr[\{u,v\} \notin \mathcal{S}, \{u,v\} \notin \mathcal{S}] \cdot \frac{1}{m}$$

$$= \sum_{(u,v) \in E} (d_u + d_v - 1)^2 \cdot \frac{1}{m^3} \sum_{(u,v) \in E} (d_u + d_v)^2$$

$$\leq \frac{2}{m^3} \sum_{(u,v) \in E} (d_u^2 + d_v^2) = \frac{3}{m^3} \sum_{e \in [n]} d_e^3.$$

For the final equality, consider the number of terms in the summation where $d_e^3$ appears. This is the number of edges $(u,v)$ (over all $v$), which is exactly $d_u$. \qed

**Claim 4.**

$$\sum_{(i,j) \in \mathcal{S}, (i,j) \cap (p,q) = \emptyset} E[X_{i,j}X_{p,q}] = 6 \binom{s}{4} \mu^2.$$  

**Proof.** There are 6 terms in the summation (\binom{4}{2} ways of choose $(i,j,p,q)$ and 6 different orderings). Note that $X_{i,j}$ and $X_{p,q}$ are independent, regardless of the structure of $G$ or the set of wedges $\mathcal{S}$. This is because $\Pr[X_{i,j} = 1 | X_{p,q} = 0] = \Pr[X_{i,j} = 1 | X_{p,q} = 1] = \Pr[X_{i,j} = 1]$. In other words, the outcomes of the random edges $r_p$ and $r_q$ do not affect the edges $r_i, r_j$ (by independence of these draws) and hence cannot affect the random variable $X_{i,j}$. Thus, $E[X_{i,j}X_{p,q}] = E[X_{i,j}]E[X_{p,q}] = \mu^2$. \qed

### 6.4 Proof of Thm. 1

**Proof.** As mentioned in the proof sketch, the output bit $b_m$ is 1 if $|\mathcal{W}_m| > 0$ and a wedge from $\mathcal{C}_m$ is sampled. For convenience, we will set $Y = |\mathcal{W}_m|$ and $Z = |\mathcal{C}_m|$. Note that $Y$ is the total number of wedges formed by edges in $\mathcal{R}_m$. And $Z$ is the number of future closed wedges formed by edges in $\mathcal{R}_m$. Both $Y$ and $Z$ are random variables that depend on $\mathcal{R}_m$. Condition on some $\mathcal{R}_m$ generated by the SINGLE-Bit. If $|\mathcal{W}_m| > 0$, then the expectation of $b_m$ is exactly $Z/Y$. The next claim extends this.

**Claim 5.** Let $\mathcal{E}$ be an event such that $\mathcal{E}$ implies $Y > 0$ and $\Pr[\mathcal{E}]$ is at least $1 - \beta'$. Then $|E[b_m] - E[Z/Y|\mathcal{E}]| \leq \beta'$.

**Proof.** Let $\mathcal{F}$ denote the event $Y > 0$. Since SINGLE-Bit outputs 0 when $\mathcal{F}$ does not hold, we get $E[b_m|\mathcal{F}] = 0$. Since $E[b_m] = E[b_m|\mathcal{F}]E[\mathcal{F}] + E[b_m|\mathcal{F}^c]E[\mathcal{F}^c]$, $E[b_m] = E[b_m|\mathcal{F}]E[\mathcal{F}]$. Further observe that $E[b_m|\mathcal{F}]$ is exactly equal to $E[Z/Y|\mathcal{F}]$. Therefore, we get $E[b_m] = E[Z/Y|\mathcal{F}]E[\mathcal{F}]$. Further conditioning the expectation on the event $\mathcal{E}$, we get

$$E[b_m] = E[Z/Y|\mathcal{F}]E[\mathcal{F}]$$

$$= E[Z/Y|\mathcal{F} \cap \mathcal{E}]E[\mathcal{F} \cap \mathcal{E}] + E[Z/Y|\mathcal{F}^c]E[\mathcal{F}^c \cap \mathcal{E}]$$

$$= E[Z/Y|\mathcal{E}]E[\mathcal{F}] + E[Z/Y|\mathcal{F}]E[\mathcal{F}^c \cap \mathcal{E}]$$

The second last equality uses the fact that $\Pr[\mathcal{A} \cap \mathcal{B}] = \Pr[\mathcal{A} | \mathcal{B}] \cdot \Pr[\mathcal{B}]$, while the last equality uses the fact that $\mathcal{F} \cap \mathcal{E} = \mathcal{E}$ (since $\mathcal{E}$ implies $\mathcal{F}$). Note that $Z/Y \leq 1$. Thus, (2) is at least

$$E[Z/Y|\mathcal{E}]E[\mathcal{F}] \geq E[Z/Y|\mathcal{E}](1 - \beta') \geq E[Z/Y|\mathcal{E}] - \beta'.$$

Moreover, (2) is at most

$$E[Z/Y|\mathcal{E}] + E[Z/Y|\mathcal{F}]E[\mathcal{F}^c \cap \mathcal{E}] \leq E[Z/Y|\mathcal{E}] + E[Z/Y|\mathcal{F}] \leq E[Z/Y|\mathcal{E}] + \beta'.$$

Henceforth, we focus on the quantity $E[Z/Y|\mathcal{E}]$ (for an appropriately chosen $\mathcal{E}$ defined shortly). The overall strategy is to show that this is close to $E[Z|E[\mathcal{E}]]$. This requires bounds for deviations of $Y, Z$ from their corresponding expectations.

First, we determine $E[Y]$ and $E[Z]$ from Lem. 1. Applying the lemma with $\mathcal{S}$ as the set of all wedges in $G$ to obtain $E[Y] = \binom{s}{2} W/m^2 = (s - 1) W/m^2$. Analogously, with $\mathcal{S}$ as the set of all future closed wedges, we get $E[Z] = \binom{s}{2} T/m^2$. (Recall that the number of future closed wedges in $G$ is $T$.)

Next we apply the second part of Lem. 1 to random variables $Y$ and $Z$ with the parameter $\beta$ in the statement of the lemma set to (the smaller quantity) $\beta' = \beta/5$. We can do this because by the premise of the theorem, $s \geq cm/(\beta^2 \sqrt{T}) \geq cm/(\beta^2 \sqrt{W})$. Moreover, for large enough constant $c$, the latter is at least $c' m/(\beta^3 \sqrt{W})$, as required in order to apply the second part of Lem. 1. Therefore, with probability $1 - \beta'$, $|Y - E[Y]| \leq \beta' E[Y]$. (This suffices to prove the second part of Thm. 1, since $E[Z] = m^2 Y/s(s - 1)$ and $m^2 E[Y]/s(s - 1) = W$.) With probability $1 - \beta'$,

$$|Z - E[Z]| \leq (\beta' W/T) E[Z].$$

Note that $(\beta' W/T) E[Z] = \beta' E[Y]$.

For convenience, let $\mathcal{E}$ denote the event

$$\max(|Y - E[Y]|, |Z - E[Z]|) \leq \beta' E[Y].$$
setting of angle counts for web-NotreDame and amazon0505. We fix a shown in of the behavior of parameters, we perform a single run of Streaming-Triangles on these streamed graphs. This is to give a true indication to 70K in increments of 10K. For each setting of the parameters, we perform a single run of Streaming-Triangles on these streamed graphs. This is to give a true indication on these streamed graphs. This is to give a true indication as-skitter 1696K 11095K 16022M 28770K 0.005 cit-Patents 3775K 16519K 336M 7515K 0.067 DBLP 317K 1049K 21M 224K 0.3064 roadNet-CA 1965K 2767K 6M 121K 0.060 web-BerkStan 685K 649K 27983M 6491K 0.007 web-Google 876K 4322K 727M 13392K 0.055 web-NotreDame 326K 1090K 305M 8910K 0.088 web-Stanford 282K 1993K 3944M 11329K 0.009 wiki-Talk 2394K 4660K 12594M 9204K 0.002 youtube 1158K 2990K 1474M 3057K 0.006 flickr 1861K 15556K 14670M 548659K 0.112 livejournal 5284K 48710K 7519M 310877K 0.124 orkut 3073K 223534K 45625M 627584K 0.041

Table 1: Properties of the graphs used in the experiments

<table>
<thead>
<tr>
<th>Graph</th>
<th>m</th>
<th>Mm</th>
<th>W</th>
<th>T</th>
<th>κ</th>
</tr>
</thead>
<tbody>
<tr>
<td>amazon0312</td>
<td>401K</td>
<td>2350K</td>
<td>69M</td>
<td>3686K</td>
<td>0.160</td>
</tr>
<tr>
<td>amazon0505</td>
<td>410K</td>
<td>2439K</td>
<td>73M</td>
<td>3951K</td>
<td>0.162</td>
</tr>
<tr>
<td>amazon0601</td>
<td>403K</td>
<td>2443K</td>
<td>72M</td>
<td>3987K</td>
<td>0.166</td>
</tr>
<tr>
<td>as-skitter</td>
<td>1696K</td>
<td>11095K</td>
<td>16022M</td>
<td>28770K</td>
<td>0.005</td>
</tr>
<tr>
<td>cit-Patents</td>
<td>3775K</td>
<td>16519K</td>
<td>336M</td>
<td>7515K</td>
<td>0.067</td>
</tr>
<tr>
<td>DBLP</td>
<td>317K</td>
<td>1049K</td>
<td>21M</td>
<td>224K</td>
<td>0.3064</td>
</tr>
<tr>
<td>roadNet-CA</td>
<td>1965K</td>
<td>2767K</td>
<td>6M</td>
<td>121K</td>
<td>0.060</td>
</tr>
<tr>
<td>web-BerkStan</td>
<td>685K</td>
<td>649K</td>
<td>27983M</td>
<td>6491K</td>
<td>0.007</td>
</tr>
<tr>
<td>web-Google</td>
<td>876K</td>
<td>4322K</td>
<td>727M</td>
<td>13392K</td>
<td>0.055</td>
</tr>
<tr>
<td>web-NotreDame</td>
<td>326K</td>
<td>1090K</td>
<td>305M</td>
<td>8910K</td>
<td>0.088</td>
</tr>
<tr>
<td>web-Stanford</td>
<td>282K</td>
<td>1993K</td>
<td>3944M</td>
<td>11329K</td>
<td>0.009</td>
</tr>
<tr>
<td>wiki-Talk</td>
<td>2394K</td>
<td>4660K</td>
<td>12594M</td>
<td>9204K</td>
<td>0.002</td>
</tr>
<tr>
<td>youtube</td>
<td>1158K</td>
<td>2990K</td>
<td>1474M</td>
<td>3057K</td>
<td>0.006</td>
</tr>
<tr>
<td>flickr</td>
<td>1861K</td>
<td>15556K</td>
<td>14670M</td>
<td>548659K</td>
<td>0.112</td>
</tr>
<tr>
<td>livejournal</td>
<td>5284K</td>
<td>48710K</td>
<td>7519M</td>
<td>310877K</td>
<td>0.124</td>
</tr>
<tr>
<td>orkut</td>
<td>3073K</td>
<td>223534K</td>
<td>45625M</td>
<td>627584K</td>
<td>0.041</td>
</tr>
</tbody>
</table>

By the union bound on probabilities, \( \Pr[\mathcal{E}] > 1 - 2\beta' \). By choice of \( s \), \( E[Y] \geq c^2/\beta'^2 \). Hence, when \( \mathcal{E} \) happens, \( Y > 0 \). (In other words, with probability at least \( 1 - 2\beta' \), the edges in \( R_w \) will form a wedge.)

Now look at \( E[Z]/Y \mathcal{E} \). From the expectation bound, we know that \( E[Z]/E[Y] = T/W = \kappa/3 \). When \( \mathcal{E} \) occurs, \( Z/Y \) is not far from this value. Formally, when \( \mathcal{E} \) occurs,

\[
E[Z] - \beta'^2 E[Y] \leq \frac{Z}{1 + \beta'^2} E[Y] \leq E[Z] + \beta'^2 E[Y] \leq \frac{1}{1 - \beta'} E[Y]
\]

Manipulating the upper bound, (for small enough \( \beta' \))

\[
E[Z] + \beta'^2 E[Y] \leq (1 + 2\beta') E[Z] + 2\beta' = \kappa/3 + 4\beta'
\]

Using a similar calculation for the lower bound, when \( \mathcal{E} \) occurs, \( |Z/Y - \kappa/3| \leq 4\beta' \). This implies \( |E[Z]/Y \mathcal{E}| - \kappa/3| \leq 4\beta' \).

Observe that the algorithm is almost always within 10% of the true answer. The statistical deviation is larger from edge reservoirs of 10K and 20K, but it quite small for 40K. Indeed for \( s_e = 40K \), the output is well within 5%. In general, these figures show that the algorithm does not have wild statistical fluctuations and is fairly well behaved as we increase the space. Increasing the space gives a predictable improvement in the estimate.

7. EXPERIMENTAL RESULTS

We implemented our algorithm in C++ and ran our experiments on a MacBook Pro laptop equipped with a 2.8GHz Intel core i7 processor and 8GB memory. The vital statistics of all the graphs that we experiment upon are provided in Tab. 1. For our case studies, we focus on the web-NotreDame and amazon0505 graphs, having 1M and 3M edges respectively.

Effects of storage on estimates: We explore the effect that the parameters \( s_e, s_w \) (referred to as the edge reservoir and wedge reservoir sizes) have on the quality of the estimates for \( \kappa \) and \( T \). Both of these graphs have around 1M edges. Each of these is converted into a random stream of edges.

We set \( s_e \) to be 10K, 20K, and 40K, and vary \( s_w \) from 10K to 70K in increments of 10K. For each setting of the parameters, we perform a single run of Streaming-Triangles on these streamed graphs. This is to give a true indication of the behavior of Streaming-Triangles. The results are shown in Fig. 3 and Fig. 4, both for the transitivity and triangle counts for web-NotreDame and amazon0505. We fix a setting of \( s_e \) and increase \( s_w \). In terms of the theory, a large \( s_e \) helps generate wedges that are closer to uniform. A larger \( s_w \) then helps to get a sharper estimate for the fraction of future closed wedges.

Comparison with previous work: The streaming algorithm of Buriol et al [9] was implemented and run on real graphs. In general, they get fairly large error even with storage of 100K edges. We provide comparisons with an implementation of their algorithm for arbitrary streams (they also have a version only for incidence streams, but we do not compare with that). The basic sampling procedure involves sampling a random edge and a random vertex and trying the complete a triangle. This is repeatedly independently in parallel to get an estimate for the number of triangles. Buriol et al provide various heuristics to speed up their algorithm, but the core sampling procedure is what was described above.

In Fig. 5, we compare runs of Streaming-Triangles with their algorithm for our test graphs. For convenience, we just refer to their algorithm as “Buriol et al”. The \( x \)-axis is the space used, and the \( y \)-axis gives the triangle estimate. For simplicity, we fix \( s_e = 20K \) and increase \( s_w \) in Streaming-Triangles. For Buriol et al, we count the storage of a single edge and vertex a single unit of space. (We ignore the extra two edges needed to complete the triangle, and simply count the number of samples used.) While the previous algorithm has estimates off by 50% or more, Streaming-Triangles is much more accurate over all of its runs. (Indeed, the figure is zoomed out so much that the statistical fluctuations of Streaming-Triangles are barely visible.) For amazon0505, the estimate give by Buriol et al is zero till 80K samples. We note that these results are consistent with those given in [9].

Runs on various graphs: We run Streaming-Triangles on a variety of graphs obtained from the SNAP database.
We simply set $s_e$ as 20K and $s_w$ as 20K for all our runs. Each graph is converted into a stream by taking a random ordering on the edges. In Fig. 6, we show our results for estimating both $\kappa$ and $T$. The absolute values are plotted for $\kappa$ together with the true values. For triangles, we plot the relative error (so $|\text{est} - T|/T$, where $\text{est}$ is the algorithm output) for each graph, since the true values can vary over orders of magnitude. Observe that the transitivity estimates are very accurate. The relative error for $T$ is mostly below 8%, and often below 4%.

All the graphs listed have millions of edges, so our storage is always 2 orders of magnitude smaller than the graph. Most dramatically, we get accurate results on the Orkut social network, which has 220M edges. The algorithm stores only 30K edges, a 0.0001-fraction of the graph. Also observe the results on the Flickr and Livejournal graphs, which also run into tens of millions of edges.

**Effect of stream ordering:** It is not possible to check that our algorithm works for all orderings of the stream. So we generate a different orderings of the edges in web-NotreDame and run STREAMING-TRIANGLES on all the orderings. The results are given in Fig. 2. As before, we fix the edge and wedge reservoir to 20K. We discuss the different orderings below.

We first have two random orderings. Next, we generate a stream through a bfs as follows. We take a bfs tree from a random vertex and list out all edges in the tree. Then, we list the remaining edges in random order. Since the average degree of web-NotreDame is around 3, the tree has about one-third of the total edges. So this ordering is fairly different from a random ordering. Our fourth ordering involves taking a dfs from a random vertex and list out edges in order as seen by the dfs. Finally, we sort the vertices by degree and list all edges incident to a vertex (this is an incidence stream).

STREAMING-TRIANGLES performs reasonably on all these different orderings. There is little deviation in the transitivity values. There is somewhat more difference in the triangle numbers, but it never exceeds 10% relative error. This seems to be simply deviations in repeated runs of STREAMING-TRIANGLES.

**Real-time tracking:** A major benefit of STREAMING-TRIANGLES is that it can maintain a real-time estimate of $\kappa_t$ and $T_t$. We take a real-world temporal graph, cit-Patents, which contains patent citation data over decades. The edges are time stamped with the year of citation, and hence give a stream of edges. Using more storage of an edge reservoir of 40K and wedge reservoir of 15K, we accurately track these values over time (refer to Fig. 1). Note that this is still orders of magnitude smaller than the full size of the graph, which is 16M edges. The errors of our estimates are overall quite small. (The true values are only given for the year ends.)

8. REFERENCES


Figure 4: amazon0505 detailed runs: We fix the edge reservoir, $s_e$, to 10K, 20K, and 40K. Then we increase the wedge reservoir size, $s_w$, from 10K to 70K and track the output of Streaming-Triangles. We give plots for transitivity and triangle counts.


[22] T. Plantenga. Inexact subgraph isomorphism in
Figure 5: Comparison of our algorithm with Buriol et al. [28] for amazon0505 and web-NotreDame. We use a 20K edge reservoir and variable wedge reservoir for our storage.

Figure 6: Output of a single run of STREAMING-TRIANGLES on a variety of real datasets with 20K edge reservoir and 20K wedge reservoir. The plot on the left gives the estimated transitivity values (labelled streaming) alongside their exact values. The plot on the right gives the relative error of STREAMING-TRIANGLES’s estimate on triangles $T$. Observe that the relative error for $T$ is mostly below 8%, and often below 4%.

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