The moral of these examples is that $\mathbb{R}^n$ contains subsets which are so strangely put together that it is impossible to define a geometrically reasonable notion of measure for them, and the remedy for the situation is to discard the requirement that $\mu$ should be defined on all subsets of $\mathbb{R}^n$. Rather, we shall content ourselves with constructing $\mu$ on a class of subsets of $\mathbb{R}^n$ that includes all the sets one is likely to meet in practice unless one is deliberately searching for pathological examples. This construction will be carried out for $n = 1$ in §1.5 and for $n > 1$ in §2.6.

It is worthwhile, and not much extra work, to develop the theory in much greater generality. The conditions (ii) and (iii) are directly related to Euclidean geometry, but set functions satisfying (i), called measures, arise also in a great many other situations. For example, in a physics problem involving mass distributions, $\mu(E)$ could represent the total mass in the region $E$. For another example, in probability theory one considers a set $X$ that represents the possible outcomes of an experiment, and for $E \subset X$, $\mu(E)$ is the probability that the outcome lies in $E$. We therefore begin by studying the theory of measures on abstract sets.

### 1.2 $\sigma$-ALGEBRAS

In this section we discuss the families of sets that serve as the domains of measures.

Let $X$ be a nonempty set. An algebra of sets on $X$ is a nonempty collection $\mathcal{A}$ of subsets of $X$ that is closed under finite unions and complements; in other words, if $E_1, \ldots, E_n \in \mathcal{A}$, then $\bigcup_1^n E_j \in \mathcal{A}$; and if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$. A $\sigma$-algebra is an algebra that is closed under countable unions. (Some authors use the terms field and $\sigma$-field instead of algebra and $\sigma$-algebra.)

We observe that since $\bigcap_1^n E_j = \left(\bigcup_1^n E_j\right)^c$, algebras (resp. $\sigma$-algebras) are also closed under finite (resp. countable) intersections. Moreover, if $\mathcal{A}$ is an algebra, then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$, for if $E \in \mathcal{A}$ we have $\emptyset = E \cap E^c$ and $X = E \cup E^c$.

It is worth noting that an algebra $\mathcal{A}$ is a $\sigma$-algebra provided that it is closed under countable disjoint unions. Indeed, suppose $\{E_j\}_{1}^{\infty} \subset \mathcal{A}$. Set

$$F_k = E_k \setminus \left[\bigcup_1^{k-1} E_j\right] = E_k \cap \left[\bigcup_1^{k-1} E_j\right]^c.$$ 

Then the $F_k$'s belong to $\mathcal{A}$ and are disjoint, and $\bigcup_1^{\infty} E_j = \bigcup_1^{\infty} F_k$. This device of replacing a sequence of sets by a disjoint sequence is worth remembering; it will be used a number of times below.

Some examples: If $X$ is any set, $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are $\sigma$-algebras. If $X$ is uncountable, then

$$\mathcal{A} = \{E \subset X : E \text{ is countable or } E^c \text{ is countable}\}$$

is a $\sigma$-algebra, called the $\sigma$-algebra of countable or co-countable sets. (The point here is that if $\{E_j\}_{1}^{\infty} \subset \mathcal{A}$, then $\bigcup_1^{\infty} E_j$ is countable if all $E_j$ are countable and is co-countable otherwise.)