Math 540 Real Analysis — Fall 2016 — Homework 0

The following policies apply to all homework assignments in this course.
(1) Students are encouraged to work together on solving the homework problems. But you must write up your own solutions, in your own words. Submission of identical solutions will give grounds for an academic integrity investigation.
(2) Homework scores will depend on the clarity of your presentation as well as the correctness of the mathematics. Use words and sentences to explain your reasoning. (By way of example, notice that the textbook is not simply a collection of formulas...)
(3) Late homework will not be accepted unless you received an extension from the professor in advance, by email.
(4) Homework assignments must be stapled in the top left corner, with your name written in the top right corner. Otherwise your homework will not be graded.
(5) Please write on the first page of your homework the names of all people with whom you discussed the problems. (This is an ethical matter: "give credit where credit is due"). This information will not affect the grading.

For extension requests and other issues about this homework assignment, please contact Professor Tyson: Tyson@illinois.edu.

Review problems on undergraduate real analysis.
To be turned in, but not for credit.

Problem 1. For $0 \leq x < \infty$ and $n \in \mathbb{N}$, let
$$f_n(x) = \frac{2n + x}{n + x}.$$ Show that $\{f_n\}$ converges uniformly on $[0, L]$ for each $L < \infty$, but $\{f_n\}$ does not converge uniformly on $[0, \infty)$.

Problem 2. Let $S$ be a subset of a metric space $(X, d)$. A point $y \in X$ is said to be a limit point of $S$ if there exists a sequence $\{x_n\}$ contained in $S \setminus \{y\}$ such that $\lim_{n \to \infty} x_n = y$.

Prove that a set $S$ is closed if and only if it contains all of its limit points.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and uniformly continuous. Prove that $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x)^2$ is uniformly continuous.

Problem 4 (Sublinearity of lim sup). Recall the definition of lim sup from [Bass] Section 1.1, and read about the extended real number system $[-\infty, \infty]$ in [Folland] Section 0.5, which is available on the “Daily Schedule” webpage.

(i) Show that if $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real numbers, then
$$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.$$ 

(ii) Find an example such that the inequality is strict.

Comment. We do not assume the sequences are bounded, and so the lim sup’s might equal $\infty$, in part (i). This fact should not affect your proof.

Incidentally, the exercise holds for all real sequences (not just nonnegative sequences) provided the right side of the inequality makes sense, that is, provided it does not have the undefined form $\infty + (-\infty)$ or $(-\infty) + \infty$. 