Math 540 Real Analysis — Fall 2016 — Homework 3

For extension requests and other issues about this homework assignment, please contact Professor Laugesen: Laugesen@illinois.edu.

**Problem 1.**
[Bass] Exercise 4.13

**Problem 2** (Each Lebesgue null set omits a copy of the rationals).
(a) [Bass] Exercise 4.18. **Hint.** $\cup_{q \in \mathbb{Q}} (q + A)$.
(b) Generalize part (a) in some way.

**Problem 3** (Uniform exterior regularity). Given a set $E$ in a metric space $X$, define the distance from a point $x \in X$ to the set $E$ by
$$\text{dist}(x, E) = \inf_{y \in E} d(x, y),$$
where $d$ denotes the metric. For each $n \geq 1$, define the open set
$$U_n = \{x \in X : \text{dist}(x, E) < 1/n\}$$
consisting of the points that lie within distance $1/n$ of $E$. Obviously $U_n \supset E$.
(a) Sketch a figure illustrating the definition of $U_n$, when $E \subset X = \mathbb{R}^2$ with the Euclidean metric.
(b) Prove that if $E$ is compact and $\mu$ is a Borel measure on $X$ that is finite on each closed ball, then $\mu(E) = \lim_n \mu(U_n)$. (Recall a Borel measure is a measure on the Borel $\sigma$-algebra.)
(c) Find an open set $E \subset (0, 1)$ for which $m(E) < \lim_n m(U_n)$. Here $X = \mathbb{R}$ and $m$ is Lebesgue measure.

**Comment.** Lebesgue measure is indeed a Borel measure, since the Lebesgue $\sigma$-algebra includes all Borel sets by [Bass] Proposition 4.9. Thus part (c) shows that part (b) can fail when $E$ is a bounded open set.

**Problem 4** (Effect of a Hölder continuous map on Hausdorff measure). Let $\alpha > 0$. A function $f : X \to Y$ is said to be $\alpha$-Hölder continuous with constant $K$ if the inequality
$$d_Y(f(x), f(x')) \leq K d_X(x, x')^\alpha$$
holds for all $x, x' \in X$. (Here $d_X$ is a metric on $X$, and $d_Y$ is a metric on $Y$.)
(a) Prove that if $f : X \to Y$ is $\alpha$-Hölder continuous with constant $K$, and $A \subset X$, then
$$\mathcal{H}^{s/\alpha, \ast}(f(A)) \leq K^{s/\alpha} \mathcal{H}^{s, \ast}(A)$$
for each $s \geq 0$. **Hint.** Show that $\text{diam}(f(E)) \leq K(\text{diam}(E))^{\alpha}$ for every set $E \subset X$.
(b) Use part (a) to prove that every 2-Hölder continuous function $f : [0, 1] \to \mathbb{R}$ is constant. **Hint.** $A = [0, 1]$ and $s = 1$.

**Comment.** Can you think of another way (not using Hausdorff measure) to prove part (b)?

**Problem 5** (Cross-section of a Borel set is a Borel set).
Fix $y \in \mathbb{R}$. Show that if $A$ is a Borel subset of $\mathbb{R}^2$ then the cross-section
$$A(y) = \{x \in \mathbb{R} : (x, y) \in A\}$$
is a Borel subset of $\mathbb{R}$.
**Problem 6** (Positive and negative part functions). Let $X$ be a set with $\sigma$-algebra $\mathcal{A}$. Given $f : X \to \mathbb{R}$, define its positive and negative part functions by

$$
 f^+(x) = \max(f(x), 0), \quad f^-(x) = \max(-f(x), 0).
$$

Show that if $f$ is measurable then $f^+, f^-$, and $|f|$ are also measurable.

**Comment.** These functions are useful in certain applications because

$$
 f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-,
$$

as you can check by considering the three cases $f(x) > 0$, $f(x) = 0$, $f(x) < 0$.

**Problem 7** (Reciprocal of a measurable function is measurable).

[Bass] Exercise 5.3

**Problem 8** (Restrictions).
Let $X$ be a set with $\sigma$-algebra $\mathcal{A}$, and suppose $f : X \to \mathbb{R}$ is measurable. Show that if $E \in \mathcal{A}$ then the function

$$
 g(x) = \begin{cases} 
 f(x) & \text{if } x \notin E, \\
 0 & \text{if } x \in E,
\end{cases}
$$

is measurable. **Hint.** One line proof. **Note.** We regard $g$ as a restriction of $f$ to $E^c$, and call $E$ the “exceptional set”.