Approximations and Graphical Derivatives

Instructions. Put the first and last name of everyone in your workgroup at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

1. Determine the linear approximation of $\sin(x)$ at $x = 0$. Use the graph of the function and its linear approximation to determine when this approximation is an overestimate of $\sin(x)$ and when it is an underestimate.

\[ f(x) = \sin(x) \quad f(0) = 0 \]
\[ f'(x) = \cos(x) \quad f'(0) = 1 \]
\[ y - 0 = f'(0) (x - 0) \]
\[ y = x \]
\[ L(x) = x \]
\[ \text{overestimates when } x > 0 \]
\[ \text{underestimates when } x < 0 \]

2. Estimate $\sqrt{8.1}$.

\[ f(x) = x^{1/3} \quad f(8) = 2 \]
\[ f'(x) = \frac{1}{3} x^{-2/3} \quad f'(8) = \frac{1}{3 \sqrt[3]{8}} = \frac{1}{12} \]
\[ y = \frac{1}{12} (x - 8) \]
\[ L(8,1) = \frac{1}{12} - (8/12) + \frac{1}{2} \]

3. A window has the shape of a square surmounted by a semi-circle. The base of the window is measured as having width 60cm with a possible error in measurement of 0.1cm. Use differentials to estimate the maximum error possible in computing the area of the window.

Area = \[ x \cdot \frac{\pi}{2} \cdot \left(1 + \frac{\pi}{8}\right) \]
\[ A(x) = (1 + \frac{\pi}{8})x^2 \]
\[ A'(x) = 2(1 + \frac{\pi}{8})x \]
\[ dA = A'(x) dx \]
\[ x = 60, \quad dx = 0.1 \]
\[ = 2 \left(1 + \frac{\pi}{8}\right)(60)(0.1) \]
\[ = 12 + 3 \pi < 16.7 \text{ cm}^2 \]
(5) Determine the exact value of \( x \) at which \( f(x) = (x^2 + 1)e^{-x} \) increases most rapidly.

\[ f'(x) = (x^2 + 1)(e^{-x}(-1)) + (2x)e^{-x} = e^{-x}[-x^2 + 1 + 2x] \]

\[ f''(x) = e^{-x}[-x^2 + 1 + 2x] \frac{d}{dx}(-x^2 + 1 + 2x) = e^{-x}(-2x - 2x + 2) \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f'(x) )</th>
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<tbody>
<tr>
<td>3</td>
<td>0</td>
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\( x = 1 \)

(6) State the Mean Value Theorem. Be sure to include IF and THEN.

**IF** \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\)

**THEN** there is a point \( c \) between \( a \) and \( b \) with

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

(7) Let \( f(x) = \frac{3x}{\ln(x)} \). Determine each interval where \( f \) is increasing and each interval where \( f \) is decreasing.

\[ f'(x) = \frac{\ln(x) - 4x}{(\ln(x))^2} \]

\( f'(x) = 0 \quad \ln(x) - 4x = 0 \quad \ln(x) = 1 \quad x = e \)

(8) The graph below is a drawing of a derivative function \( f'(x) \). From this graph, read off the intervals where \( f(x) \) is increasing/decreasing, and list the x-coordinates where \( f(x) \) has a maximum or minimum (identify which one it is).

\( f(x) \) increasing on \((-1, 0) \cup (2, 4)\)

\( f(x) \) decreasing on \((0, 2) \cup (4, 5)\)

Critical Points: \( x = 0, 2, 4 \)

Local Min: \( x = 2 \)

Local Max: \( x = 0, 4 \)

(9) The function below is a graph of \( f'(x) \). Use this graph to determine all inflection points of \( f(x) \), and the intervals where \( f(x) \) is concave up (ccu) or concave down (ccd).

Inflection Points: \( x = 0, 2.25, 3.8, 5.5 \)

ccu \((-1, 0) \cup (2.25, 3.8) \cup (5.5, 6)\)

ccd \((0, 2.25) \cup (3.8, 5.5)\)