4. **Defn.** \[ f \text{ is continuous (cont.) at } a \]

point \( x = a \) means

1) \( \lim_{x \to a} f(x) \) exists (not \( \pm \infty \))

2) \( f(a) \) defined

and 3) \( \lim_{x \to a} f(x) = f(a) \)

II. **Defn.** \( f \text{ is cont. from the left at } x = a. \)

means

1) \( \lim_{x \to a^-} f(x) \) exists (not \( \pm \infty \))

2) \( f(a^-) \) defined

3) \( \lim_{x \to a^-} f(x) = f(a) \)

R- cont. replace "-" by "+" above 

III. **Defn.** \( f \text{ is cont. on an interval } I \)

if \( f \) is cont. at every point of \( I \).
<table>
<thead>
<tr>
<th>#</th>
<th>(\lim_{x \to a^-} f(x))</th>
<th>(\lim_{x \to a^+} f(x))</th>
<th>(\lim_{x \to a} f(x))</th>
<th>(f(a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2/3</td>
<td>2/3</td>
<td>DNE</td>
<td>undefined</td>
</tr>
<tr>
<td>2</td>
<td>2/3</td>
<td>2/3</td>
<td>DNE</td>
<td>undefined</td>
</tr>
<tr>
<td>3</td>
<td>(\infty)</td>
<td>(\infty)</td>
<td>(\infty)</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>DNE</td>
<td>DNE</td>
<td>DNE</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>L-cont</td>
<td>DNE</td>
<td>DNE</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>DNE</td>
<td>3</td>
<td>DNE</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>R-cont</td>
<td>(\infty)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Types of discontinuities

**Removable discontinuity**

\[
\lim_{x \to a} f(x) \text{ exists (not } \pm \infty) \]

- can remove discontinuity by redefining \( f(a) \).

**Jump discontinuity**

Both \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) exist (not \( \pm \infty \))

- but not equal to each other.

- \( f(a) \) may or may not be defined.

**Note:** There are other types of (unnamed) discontinuities:

- **Infinite Discontinuity**
  
  \[
  \lim_{x \to a^+} f(x) \text{ or } \lim_{x \to a^-} f(x) = \pm \infty \]
  
  - \( a \to \infty \)

- **Other Type of Disc:**
  
  \[
  \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \]
• Combining continuous functions

If \( f \) and \( g \) are both cont. at \( a \),

\[
\begin{align*}
f \pm g \\
f - g \\
f \cdot g \\
cf \\
\frac{f}{g}
\end{align*}
\]

all cont. at \( a \)

\( \text{constant} \)

\( \frac{f}{g} \)

\[ g(a) \neq 0 \]

(Follows from limit laws. => we get)

E.g. \( \lim_{x \to a} f(x) = L, \lim_{x \to \infty} g(x) = M \)

\[
\begin{align*}
\lim_{x \to a} (f(x) + g(x)) &= L + M \\
\lim_{x \to a} cf(x) &= cL \\
\lim_{x \to a} f(x)g(x) &= \lim_{x \to a} f(x) \lim_{x \to a} g(x) = LM \\
\lim_{x \to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M} \text{ if } M \neq 0
\end{align*}
\]
Continuous functions at all $x$ except as noted:

- Polynomials $p(x)$

- Rational functions where denominator $= 0$ except where $q(x) = 0$

- $\sin x$, $\cos x$

- $\tan x$, $\cot x$, $\sec x$, $\csc x$ except where they are undefined

- $e^x$

- $\ln x$ (where defined)

- Absolute value

- Roots (where defined) $\sqrt[n]{f(x)} > 0$

- Inverse trig (where defined)
Composition of functions, example:

Where is \( f(x) = \ln \left( \frac{1}{x^2 - 1} \right) \) continuous?

**Answer:** When \( x < -1 \) and when \( x > 1 \).

**First:** \( \frac{1}{x^2 - 1} \) is cont. where it is defined. At all \( x \) except \( 1, -1 \).

**Next:** \( \ln y \) is cont. where it is defined, i.e. where \( y > 0 \); here \( y = \frac{1}{x^2 - 1} > 0 \) for \( x < -1 \) or \( x > 1 \).

\[ f(x) = \ln \left( \frac{1}{x^2 - 1} \right) \] is cont. where defined, i.e., for \( x < -1 \) or \( x > 1 \).

\[ \text{Takeaway} \]

**fog**

\( (f \circ g)(x) \) is cont. at \( x = a \) if

1. \( g(x) \) is cont. at \( x = a \) and if
2. \( f(x) \) is cont. at \( g(a) = x \)
IVT:

Suppose \( f \) is cont. on \([a,b]\) and \( N \) is a number

1) between \( f(a) \) and \( f(b) \)

2) \( f(a) \neq f(b) \)

(open interval)

C don't know if \( f(a) \leq f(b) \) or \( f(b) \leq f(a) \), so we write it this way.

\( N \in (f(a), f(b)) \) or \( N \in (f(b), f(a)) \)

Then there exists a number \( c \in (a,b) \) with \( f(c) = N \).

Graphically e.g.: \( f(b) \), \( f(a) \), \( N \).

Think "horizontal lines".

\( f(c) = N \), guaranteed if \( f \) cont. on \([a,b]\).

App: finding zeroes of functions. (Later)
Ex (IVT) Prove the equation \( \sin x = 0.72 \) has at least one solution.

**Solution:** \( f(x) = \sin x \) is continuous everywhere \( \Rightarrow \) we may apply IVT.

**DEA:** choose an interval where we think a solution exists. The desired value \( M = 0.72 \) lies between the two function values.

\[ \sin 0 = 0 \quad \text{and} \quad \sin \frac{\pi}{2} = 1 \]

Notice \( 0.72 \in [0, 1] = [f(a), f(b)] \) and so by IVT, there exists \( c \in (0, \frac{\pi}{2}) \) such that \( f(c) = 0.72 = M \).

I.e. \( \sin c = 0.72 \)

And so the boxed equation has a solution.
COROLLARY to IVT:
"Existence of Zeros":

\( \text{If } f(a) \neq 0 \text{ AND } f(b) \neq 0 \text{ AND } f(a), f(b) \text{ have opposite signs (i.e. one is } > 0, \text{ one is } < 0 \) \)

THEN
\( f(x) \text{ has a zero on } (a, b) \).

Ex. Show that \( \cos x = x \) has a soln on the interval \([0, 1]\).

\[ \text{Solution: } \quad \cos x = x \text{ has a soln on } [0, 1] \iff \cos x - x \text{ has a root on } [0, 1]. \]

Let \( f(x) = \cos x - x \).
\( f(x) \) is cont \( \checkmark \)

\( f(0) = 1 \)
\( f(1) = 0.46 \)

\( \therefore \) \( \exists c \in (0, 1) \) such that \( f(c) = 0 \)
Therefore, since $0 \in [f(1), f(0)]$,

$\Rightarrow$ there exists $c \in (0, 1)$ such that $f(c) = 0$ by IVT.

1.3. $\cos c - c = 0$
1.3. $\cos c = c$ for $c \in (0, 1)$ as required.
Idea:

\[ f(t) = \text{temperature} \]
\[ f(t_a) = 80^\circ \text{ noon yesterday} \]
\[ f(t_b) = 75^\circ \text{ now} \]

Since 78° is between 75° and 80°,
at some time \( t_c \) in between \( t_a, t_b \) the temperature 78° must have been achieved:

1.3. \( f(t_c) = 78^\circ \text{ some } t_c \)
**Definition**

Irrational: Suppose \( f(x) \) is defined on \((a, \infty)\).

\( \lim_{x \to \infty} f(x) = L \) means that the values of \( f(x) \) can be made as close to \( L \) as we'd like by taking \( x \) suff. large.

Graphically, for \( x > c \), \( f(x) \)-vals w’ll in dashed line of \( L \).

**Example**

\[
\lim_{x \to \infty} \frac{3x^3 - 3x + 2}{2x^3 + 4x^2}
\]

**Technique:** Find the highest power in the denominator, and "pull" it out of the top/bottom of the fraction.

\[
= \lim_{x \to \infty} \frac{3 - \frac{3}{x^2} + \frac{2}{x^3}}{2 + \frac{4}{x}} \\
= \lim_{x \to \infty} 3 - \frac{3}{x^2} + \frac{2}{x^3} \quad \cdot \quad \frac{x^3}{x^3}
\]

\[
= \lim_{x \to \infty} 3 - \frac{3}{x^2} + \frac{2}{x^3}
\]

\[
= \lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{3}{x^2} + \lim_{x \to \infty} \frac{2}{x^3}
\]

As we go off toward positive infinity, we have a \( \text{HA} \) at \( y = \frac{3}{2} \).

\[
\text{Defn: If either } \lim_{x \to \infty} f(x) = L \quad \Rightarrow \text{the (horiz) line} \\
\text{OR } \lim_{x \to \infty} f(x) = \infty \quad y = L, y = M \quad \text{as a } \text{HA}.
\]

What about \( \lim_{x \to -\infty} f(x) \) here?

In this case, we get the same answer: \( \lim_{x \to -\infty} f(x) = \frac{3}{2} \) (\( \Rightarrow \text{HA of } f(x) \text{ is } y = \frac{3}{2} \))
Pass the Class

plot \( \frac{3x^3 - 3x + 2}{2x^3 + 4x^2} \)

Input interpretation:

\[
\text{plot} \quad \frac{3x^3 - 3x + 2}{2x^3 + 4x^2}
\]

Plots:

\[
\begin{align*}
\text{(x from 2.1 to 5.3)} \\
\text{(x from -17 to 20)}
\end{align*}
\]
\[ \lim_{x \to \infty} \frac{15x^3 + 1}{\sqrt[3]{4x^6 + 2x + 5}} = \infty \]

\[ \lim_{x \to \infty} \frac{15x^3 + 1}{\sqrt[3]{4 + \frac{2}{x^5} + \frac{5}{x^6}}} = \infty \]

\[ \lim_{x \to \infty} \frac{15 - \frac{1}{3}x^3}{\sqrt[3]{4 + \frac{2}{x^5} + \frac{5}{x^6}}} = \frac{15 - 0}{\frac{1}{\sqrt[3]{4}}} = \frac{15}{\frac{1}{\sqrt[3]{4}}} = \frac{15}{\sqrt[3]{4}} \]

The highest power inside the square root of the denominator is \(x^6\), which simplifies to \(\sqrt[3]{x^6} = x^2 = (x^3)^2\) after the square root is applied. (but \(x \to +\infty\))

What about \(\lim f(x) \) here?

\[ \lim_{x \to -\infty} \frac{15x^3 + 1}{\sqrt[3]{4x^6 + 2x + 5}} = -\infty \]

\[ \lim_{x \to -\infty} \frac{15x^3 + 1}{\sqrt[3]{4 + \frac{2}{x^5} + \frac{5}{x^6}}} = -\infty \]

\[ \lim_{x \to -\infty} \frac{15 - \frac{1}{3}x^3}{\sqrt[3]{4 + \frac{2}{x^5} + \frac{5}{x^6}}} = \frac{15 - 0}{\frac{1}{\sqrt[3]{4}}} = \frac{15}{\frac{1}{\sqrt[3]{4}}} = \frac{15}{\sqrt[3]{4}} \]

\(\Rightarrow\) H.A.

of \(f(x)\) are \(y = -\frac{15}{2}, y = \frac{15}{2}\).

Graph it using Wolfram alpha!!!
To find all HA of a fcn \( f(x) \), compute \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

\[ \text{EX} \] Find all HA of \( f(x) = \frac{42 + 12e^{12x}}{9e^{4x} + 6} \)

Notice: The largest power in the denominator is \( e^{4x} \)

Note! Always "brand" the limit first.

\[ \lim_{x \to \infty} \frac{42 + 12e^{12x}}{9e^{4x} + 6} = \lim_{x \to \infty} \frac{\frac{42}{e^{4x}} + 12e^{8x}}{9 + \frac{6e^{4x}}{e^{4x}}} = \frac{0 + \infty}{9 + 0} = +\infty \]

No HA to the right \((k > 1)\)

Notice \( \lim_{x \to \infty} e^{kx} = +\infty \)

\( \lim_{x \to -\infty} e^{kx} = 0 \)
\[ \lim_{x \to -\infty} \frac{42 + 12e^{12x}}{9e^{4x} + 6} \to \frac{42}{6} = 7 \]

Note: Always "find" the limit first!

\[ = \lim_{x \to -\infty} 42 + \lim_{x \to -\infty} 12e^{12x} \]

\[ = \lim_{x \to -\infty} 9e^{4x} + \lim_{x \to -\infty} 6 \]

\[ = \frac{42 + 0}{0 + 6} = \frac{42}{6} = 7 \]

\( f(x) \) has a HA of \( y = 7 \)
(on the left side of the graph)
This limit involves an indeterminate form as $x \to \infty$. To resolve this, we multiply by the conjugate. Here's the solution:

\[
\lim_{{x \to \infty}} \frac{\sqrt{ax^2 + x} - 3x}{\sqrt{ax^2 + x} + 3x}
\]

\[
= \lim_{{x \to \infty}} \frac{\sqrt{ax^2 + x} - (3x)}{\sqrt{ax^2 + x} + 3x} \cdot \frac{\sqrt{ax^2 + x} + 3x}{\sqrt{ax^2 + x} + 3x}
\]

\[
= \lim_{{x \to \infty}} \frac{(ax^2 + x) - (3x)^2}{\sqrt{ax^2 + x} + 3x}
\]

\[
= \lim_{{x \to \infty}} \frac{x}{\sqrt{x^2(\frac{a}{x} + \frac{1}{x^2})} + 3x}
\]

\[
= \lim_{{x \to \infty}} \frac{1}{\sqrt{a + \frac{1}{x}} + 3}
\]

\[
= \frac{1}{\sqrt{a} + 3}
\]