Lecture #5

* Check class webpage for "Derivatives Handout"

- Memorize all rules on handout immediately, though we will slowly prove/explain them over the next 2 weeks.

- We will use all the different notations listed.

Review

The slope of the tangent line to \( y = f(x) \) at the point \( x = a \)

\[
f'(a) = m_{\text{tan}} = \lim_{{x \to a}} \frac{f(x) - f(a)}{x - a}
\]

\[
= \lim_{{h \to 0}} \frac{f(a+h) - f(a)}{h}
\]

The derivative of \( y = f(x) \) at \( x = a \)

A formula for the slope of the tangent line at an arbitrary point "x"

\[
f'(x) = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h}
\]

This outputs a number, \( h \to 0 \) outputs a function.
\[ f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \]

Find \( f'(x) \)

If \( x > 0 \), then \((x+h) > 0\) for small \( h \) (we can choose \( h \) small enough so)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
= \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{x+h-x}{h} \\
= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = \boxed{1} \]

If \( x < 0 \), then \((x+h) < 0\) for small \( h \) (we can choose \( h \) small enough so)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h} \\
= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h} \\
= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = \boxed{-1} \]
What about $x = 0$?

\[
f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}
\]

Right-hand limit
\[
\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} h = \lim_{h \to 0} 1 = 1
\]

So
\[
\lim_{h \to 0} |h| = \text{DNE}
\]

Left-hand limit
\[
\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0} -1 = -1
\]

for $f(x) = |x|

\[
f'(x) = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 \\
\text{DNE} & \text{if } x = 0 
\end{cases}
\]

Geometrically, we see there is no unique tangent line at $f(x) = |x|$ at $x = 0$.
**Defn.** If is differentiable at \( x = a \) if \( f'(a) \) exists.

- If is differentiable on an interval if \( f \) is differentiable at every point of the interval.

In our previous example, \( f(x) = |x| \) is not differentiable at \( x = 0 \) is differentiable on \( (-\infty, 0) \) and \( (0, \infty) \).
Let's look at the graphs of some functions and their derivatives (which we previously calculated).

\[ y = x^3 \quad y' = 3x^2 \]

The y-value outputted is the slope of the tangent line of the previous graph.

At the point \( x = 0 \), the function \( y = |x| \) is continuous but not differentiable. There is no output in the derivative function.

\[ y = x \quad y = x \]

\[ y = 0 \quad y = -1 \]

\[ y = 0 \quad y = 1 \]

It is not connected to the slope of the tangent line in this graph.
At $x=0$, the function $y = \frac{1}{x}$ is not continuous or differentiable.

The $y$-value outputted is the slope of the tangent line of the previous graph.

It is not connected with the slope of the tangent line of this graph.

Is there a graph that is not continuous at $x=a$?

Turns out, this is not possible. How could you prove no such graph exists?
Thm. If \( f(x) \) is differentiable at \( x = a \) then \( f(x) \) is continuous at \( x = a \).

**Proof Ideas**

Notice that differentiable and continuous both mean certain limits exist.

\[
\begin{align*}
\text{\( f(x) \) is differentiable at \( x = a \) means} & \\
& \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{this limit exists}
\end{align*}
\]

\[
\begin{align*}
\text{\( f(x) \) is continuous at \( x = a \) means} & \\
& \lim_{x \to a} f(x) = f(a)
\end{align*}
\]

i.e.

\[
\begin{align*}
\lim_{x \to a} [f(x) - f(a)] &= 0
\end{align*}
\]

Use algebra to show that if the first limit exists, the second limit is 0.
We can use the definition of derivative to find patterns in the derivatives of different types of functions.

1. \( f(x) = c \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0
\]

\( f'(x) = 0 \)

2. \( f(x) = x \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x+h)-(x)}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1
\]

\( f'(x) = 1 \)

{Not Shown in Class}
Review the expansion of \((a+b)^n\) using Pascal’s \(\Delta\) below

\[
f(x) = x^n
\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}
\]

\[
= \lim_{h \to 0} \frac{x^n + n\cdot x^{n-1}h + \text{higher order terms}}{h}
\]

\[
= \lim_{h \to 0} \frac{n\cdot x^{n-1}h + \text{all terms have a factor of } h^2 \text{ or higher}}{h}
\]

\[
= \lim_{h \to 0} n\cdot x^{n-1}
\]

\[
f'(x) = n\cdot x^{n-1}
\]

\[
\begin{array}{c}
1 \\
1 2 1 \\
1 3 3 1 \\
1 4 6 4 1 \\
1 5 10 10 5 1
\end{array}
\]

\[
(a+b)^2 = a^2 + 2ab + b^2
\]
\[
(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3
\]
\[
(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\]

Note: \(f(x) = x^n\), \(f'(x) = n\cdot x^{n-1}\)

holds for any \(x\)

(though this proof is only for \(n=\) positive integer)
4) \( f(x) = g(x) + h(x) \)

\[
\begin{align*}
    f'(x) &= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \\
    &= \lim_{h \to 0} \left[ g(x+h) + h(x+h) - g(x) - h(x) \right] \\
    &= \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h \to 0} \frac{h(x+h)-h(x)}{h} \\
    &= g'(x) + h'(x)
\end{align*}
\]

\[ f'(x) = g'(x) + h'(x) \]

**EX:** Calculate \( \frac{d}{dx} \left( x^e + e^x \right) \)

\[
\begin{align*}
    &= \frac{d}{dx} (x^e) + \frac{d}{dx} (e^x) \\
    &= e \cdot x^{e-1} + e^x
\end{align*}
\]