**Motivating Problem.** Find a practical method for calculating $e^x, \sin(x), \ldots$ Often you have an accuracy in mind (e.g., “to five decimal places”).

Taylor polynomials/series give you a way to estimate the value of a function $f$ near a real number $a$, if you know the derivatives of $f$ at $a$.

Let $f$ be a function, and let $a$ be a real number. Let $n \geq 0$ be an integer.

**Definition 1.** The *degree* $n$ (“Taylor polynomial”) approximation to $f$ at $a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$  

**Example 2.**

$$T_1(x) = f(a) + f'(a)(x-a)$$

is called the “linear approximation to $f$ at $a$.” In the following picture, the blue curve is the graph $y = f(x)$. The red line is the tangent line at $(a, f(a))$. It has slope $f'(a)$, so by the point-slope method its equation is

$$y - f(a) = f'(a)(x-a)$$

which you could also write

$$y = f(a) + f'(a)(x-a).$$

That’s $T_1$. 

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*Date: August 2009.*
Properties of $T_n$. Here are some important properties of $T_n$.

a) $T_n$ is a polynomial of degree at most $n$
   [usually the degree is $n$, but it is less than $n$ if $f^{(n)}(a)$ happens to equal 0]

b) $T_n^{(k)}(a) = f^{(k)}(a)$ if $k \leq n$
   [so $T_n$ has the same derivatives as $f$ at $x = a$, up to order $n$]

c) $T_n^{(k)}(a) = 0$ if $k > n$
   [taking more than $n$ derivatives gives zero, because $T_n$ has degree at most $n$]

Let’s check this last fact.

$$T'_n(x) = f'(a) + \frac{f''(a)}{2!} 2(x - a) + \frac{f'''(a)}{3!} 3(x - a)^2 + \text{H.O.T.} \quad \text{("Higher Order Terms")}.$$  

“Higher order terms” means terms involving $(x - a)^k$ with $k > 2$. Substituting $x = a$ shows

$$T'_n(a) = f'(a).$$

For the second derivative,

$$T''_n(x) = f''(a) + \frac{f'''(a)}{3!} 3 \cdot 2(x - a) + \text{H.O.T.}$$

where here H.O.T. means terms involving $(x - a)^k$ with $k > 1$. Substituting $x = a$,

$$T''_n(a) = f''(a).$$

What happens at the next stage...?

**Example 3.** $f(x) = e^x$, $a = 0$. Here’s a table of derivatives of $e^x$ at 0:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$f^{(k)}(x)$</th>
<th>$f^{(k)}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$e^x$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$e^x$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$e^x$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$e^x$</td>
<td>1</td>
</tr>
<tr>
<td>$n$</td>
<td>$e^x$</td>
<td>1</td>
</tr>
</tbody>
</table>

So

$$T_n(x) = 1 + 1(x - 0) + \frac{1}{2!}(x - 0)^2 + \frac{1}{3!}(x - 0)^3 + \cdots + \frac{1}{n!}(x - 0)^n$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$  

**Example 4.** $f(x) = \sin(x)$, $a = 0$, $n = 5$. Here’s a table of derivatives of $\sin(x)$ at 0:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$f^{(k)}(x)$</th>
<th>$f^{(k)}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\sin x$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\cos x$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$- \sin x$</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$- \cos x$</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>$\sin x$</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>$\cos x$</td>
<td>1</td>
</tr>
</tbody>
</table>
(there’s a pattern: every fourth line is the same). For example,
\[ T_5(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}. \]
Notice there are only odd powers of \( x \): \( T_5 \) is odd, just like the sine function.

**Remark 5.**

\[ T_2(x) = 0 + x + 0 = x \]
so we say that “\( \sin x \) equals \( x \) to degree 2, or second order”

**Example 6.** \( f(x) = \cos x, a = 0, n = 6 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( f^{(k)}(x) )</th>
<th>( f^{(k)}(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \cos x )</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( -\sin x )</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( -\cos x )</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>( \sin x )</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>( \cos x )</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>( -\sin x )</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>( -\cos x )</td>
<td>-1</td>
</tr>
</tbody>
</table>

Again, every fourth line is the same. We have
\[ T_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6. \]

**Remark 7.** We say that “\( \cos x = 1 - x^2/2 \) to third order”, since \( T_3(x) = 1 - x^2/2 \).

**Example 8.** \( f(x) = 1/x = x^{-1}, a = 1, n = 4 \). This time we cannot use \( a = 0 \). (Why not?!) Here’s a table of derivatives of \( 1/x \) at 1:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( f^{(k)}(x) )</th>
<th>( f^{(k)}(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( x^{-1} )</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( -x^{-2} )</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>( 2x^{-3} )</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>( -3!x^{-4} )</td>
<td>-3!</td>
</tr>
<tr>
<td>4</td>
<td>( 4!x^{-5} )</td>
<td>4!</td>
</tr>
</tbody>
</table>

We have
\[ T_4(x) = 1 - 1(x - 1) + \frac{2}{2!}(x - 1)^2 - \frac{3!}{3!}(x - 1)^3 + \frac{4!}{4!}(x - 1)^4 \]
\[ = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4. \]
Series. For many functions \( f \), and many choices of \( a \) and \( x \), it turns out that \( T_n(x) \rightarrow f(x) \) as \( n \rightarrow \infty \). Then we say the “Taylor series converges to \( f(x) \).” We will learn later how mathematicians make sense of convergence of series with infinitely many terms. For now, we just state how our examples behave.

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all } x
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad \text{for all } x
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad \text{for all } x
\]

\[
\frac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots = \sum_{k=0}^{\infty} (-1)^k (x - 1)^k \quad \text{for } 0 < x < 2
\]

The last series for \( 1/x \) obviously cannot be valid at \( x = 0 \). It is also false at \( x = 2 \) (try it!).

Four things you can do with series.

1. **Numerical approximation.** Use a degree two (also called “second order” or “quadratic”) Taylor polynomial to estimate \( \cos(0.1) \).

   We’ll use \( a = 0 \) since this is near to \( x = 0.1 \) and we know the Taylor polynomial for cosine when \( a = 0 \). We find

   \[
   \cos(0.1) \approx 1 - \frac{(0.1)^2}{2} = 1 - 0.005 = 0.995.
   \]

2. **Find new series.** Find the sixth-order Taylor polynomial for \( x \sin(2x) \) at 0.

   We won’t calculate afresh all the derivatives of this function at 0. Instead we take the series for \( \sin x \), then substitute \( 2x \) in place of \( x \), and multiply the whole series by \( x \):

   \[
   x \sin(2x) = x \left( 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \text{H.O.T.} \right)
   = 2x^2 - \frac{8x^4}{3!} + \frac{32x^6}{5!} + \text{H.O.T.}
   \]

Exercises.

(a) Find the series for \( e^{-x^2} \), when \( a = 0 \).

(b) Find the first few terms of the series for \( e^{-x^2} \cos x \), when \( a = 0 \).
3. Find new equations. Remember that $i$ is the complex number with $i^2 = -1$. So $i^3 = -i$ and $i^4 = 1$. Then

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \text{H.O.T.}$$

$$= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \text{H.O.T.}$$

Let’s compare this to the series for $\cos(x) + i \sin(x)$. We have

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \text{H.O.T.}$$

and

$$i \sin x = ix - \frac{x^3}{3} + \text{H.O.T.}$$

It turns out that indeed $e^{ix} = \cos x + i \sin x$. This is called “De Moivre’s Theorem.”

4. Graphical approximation. We conclude with some illustrative graphs of polynomial approximations to $\sin x$ at $a = 0$. (You might want to try the same thing yourself, with the cosine, or with $e^x$, or your favorite function.)

$$T_1(x) = x$$
$$T_3(x) = x - \frac{1}{3!}x^3$$
$$T_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$
$$T_7(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$
This is the linear approximation, $T_1$. The graph of sin is blue, the linear approximation is red. Graphically, $\sin x \simeq T_1(x) = x$ when $x$ is near $a = 0$ (say, when $|x| \lesssim \pi/6$).

This is the third-degree approximation, $T_3$. Again the graph of sin is blue, the approximation is red. Graphically, $\sin x \simeq T_3(x) = x - x^3/3!$ when $x$ is near $a = 0$ (say, when $|x| \lesssim \pi/3$).
Fifth degree.

Seventh degree.
Just for fun, here’s the degree-13 approximation. The red plot is the approximation. Notice that we finally have 5 roots, but after that the approximation blows up.

What you cannot do with Taylor series. The trigonometric meaning of sine and cosine is hidden by the Taylor series. For example, putting $x = \pi/2$ into the series for $\cos x$ must give $\cos \pi/2 = 0$, but one cannot see that result just from looking at the series!