Approximate Integration

\[\Delta x = \overline{x}_i = x_i = \]

- **Midpoint Rule**
  \[
  \int_a^b f(x)dx = \Delta x [f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)]
  \]

- **Trapezoidal Rule**
  \[
  \int_a^b f(x)dx = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]
  \]

- **Simpsons Rule**
  \[
  \int_a^b f(x)dx = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)]
  \]

Arc Length

- Write the formula in terms of \( x \): \( y = f(x), a \leq x \leq b \)
  \[
  L = \int_a^b \sqrt{1 + f'(x)^2} \, dx
  \]

- Write the formula in terms of \( y \): \( x = g(y), c \leq y \leq d \)
  \[
  L = \int_c^d \sqrt{1 + g'(y)^2} \, dy
  \]
Area of a Surface of Revolution

General formula:

\[ S = \int 2\pi R \, ds \]

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<th>rotate about x-axis [(R=\quad )]</th>
<th>integral in terms of x</th>
<th>integral in term of y</th>
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<tr>
<td>rotate about y-axis [(R=\quad )]</td>
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**Example:**
Set up an integral for the area of the surface obtained by rotation the curve \( y = \tan(x), 0 \leq x \leq \pi/3 \)

- about the x-axis in terms of x:

- about the x-axis in terms of y:

- about the y-axis in terms of x:

- about the y-axis in terms of y:
Hydrostatic force

General formula:

\[ F = \int_{a}^{b} \rho g \text{(depth)} \text{(width)} \, dy \]

1. Rectangle:

2. Triangle:

3. Parabola, circle, ellipse etc:
Moments and Centers of Mass

\[ M_y = \rho \int_a^b \, dx \]

\[ M_x = \rho \int_a^b \, dx \]

\[ \bar{x} = \]

\[ \bar{y} = \]

What if the region lies between two curves \( y = f(x) \) and \( y = g(x) \) where \( f(x) \geq g(x) \)?

\[ M_y = \]

\[ M_x = \]

Example: A lamina of density \( \rho \) kg/m\(^2\) has the shape of the half circle defined by

\[ x^2 + y^2 = 9, \quad y \geq 0. \]

Set up but do not evaluate an integral to compute the moment \( M_x \) about the x-axis.
Sequences

If \( \lim_{n \to \infty} a_n \) exists (as a finite number), we say the the sequence \( \{a_n\} \). Otherwise we say that the sequence is .

Examples: Are the following sequences convergent or divergent?

a. \( a_n = \frac{\sqrt{9n^2 - 2n}}{2n + 3} \)

b. \( a_n = \ln(n + 6) - \ln(n) \)

c. \( a_n = \frac{\cos^2 n}{4^n} \)
Series

Given a series \( \sum_{n=1}^{\infty} a_n \), let \( s_n \) denote its \( n \)th partial sum:

\[
\begin{align*}
  s_n & = \\
\end{align*}
\]

If the sequence \( \{s_n\} \) is convergent, then the series \( \sum_{n=1}^{\infty} a_n \) is called \( \) and we write

\[
\begin{align*}
  \sum_{n=1}^{\infty} a_n & = \\
\end{align*}
\]

- How can we find \( a_n \) if \( s_n \) is given?

\[
\begin{align*}
  a_n & = \\
\end{align*}
\]

- Examples of series we know well:

  **The geometric series**

  \[
  \sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + \ldots
  \]

  is convergent/divergent if \( |r| < 1 \) and its sum is:

  If \( \) , the geometric series is **divergent**.

  **p-series**
Tests we can use to find convergence or divergence:

**Test for Divergence**

**The Integral Test**
- What are the hypothesis for $f$?
- What is the conclusion?

**The Comparison Test**
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also divergent.

**The Limit Comparison Test**
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n \to \infty} \frac{a_n}{b_n} = L$, then either both series converge or both diverge.

Remarks:
- Careful when using the divergence test: If $\lim_{n \to \infty} a_n = 0$, we cannot conclude anything from the divergence test.
  Example: Look at $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Both limits are 0, but the first one diverges and the second one converges.
- If using the Integral test make sure you check the hypothesis.
Example: Use the integral test to show whether $\sum_{n=1}^{\infty} e^{-n}$ converges:

Examples: Are the following series convergent or divergent?

a. $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{2n^2 + 3}$

b. $\sum_{n=1}^{\infty} \arctan(n)$

c. $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{3^n}$

d. $\sum_{n=1}^{\infty} \sin\left(\frac{4}{n}\right)$

e. $\sum_{n=1}^{\infty} \frac{n^3 + 5n}{e^n}$
Remember: \( \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \ldots \) and we define

\[
S_n = a_1 + \cdots + a_n \quad \text{and} \quad R_n = a_{n+1} + a_{n+2} + \ldots
\]

When we approximate \( \sum_{n=1}^{\infty} a_n \) by \( S_n \) we make an "error" \( R_n \) and we want to know how big this error is.

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<th>Reminder Estimate for the Integral Test</th>
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\[
\leq R_n \leq \sum_{n=1}^{\infty} a_n \leq S
\]

- How many terms of the series \( \sum_{n=1}^{\infty} \frac{5}{n^3} \) would we need to add to estimate the sum to within 0.1?

- Approximate \( \sum_{n=1}^{\infty} \frac{5}{n^3} \) within 0.1.