Mathematical models from last lecture:

1. **Population dynamics.** $\frac{dp}{dt} = kp - r$. $p =$ population; $k =$ net growth rate; $r =$ predation $(k, r > 0)$.

2. **Falling object.** $\frac{dv}{dt} = g - \frac{\gamma}{m}v$. $v =$ downward velocity; $t =$ time; $g =$ free fall acceleration ($9.8$ m/s$^2$); $m =$ mass; $\gamma =$ air resistance constant $(\gamma, m > 0)$.

We solved these differential equations qualitatively using direction fields.

Next goal: solve the differential equations quantitatively.

**FIRST ORDER, LINEAR, CONSTANT COEFFICIENT DIFFERENTIAL EQUATION**

Solve $\frac{dy}{dt} = ay - b = a(y - \frac{b}{a})$ (constant $a \neq 0$, constant $b$).

1) Equilibrium solution: $y(t) = \frac{b}{a}$ (check $\frac{dy}{dt} = 0 = ay - b$).

2) Non-equilibrium solutions: if $y(t) \neq \frac{b}{a}$ then $\frac{dy/dt}{y - b/a} = a$. Integrate (antidifferentiate) both sides:

$$\ln |y - \frac{b}{a}| = at + K,$$

where $K$ is a constant. (Check by differentiating both sides w.r.t. $t$, using chain rule on left side.)

Exponentiate: $|y - \frac{b}{a}| = e^{at+K} = e^K e^{at}$.

Eliminate absolute value: $y - \frac{b}{a} = Ce^{at}$, with $C = \pm e^K$.

Memorize the general solution: $y = \frac{b}{a} + Ce^{at}$ Check DE: $dy/dt = aCe^{at}$ and $ay - b = aCe^{at}$.

Different values of $C$ give different solution curves. Determine $C$ from the initial condition $y(0) = y_0$. Plug in $t = 0$: $\frac{b}{a} + C = y_0$, hence $C = y_0 - \frac{b}{a}$. Conclude the initial value problem

$\frac{dy}{dt} = ay - b, \quad y(0) = y_0$

has solution

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}.$$ 

**APPLICATIONS**

1. **Mice.** Initial value problem: $\frac{dp}{dt} = kp - r$, $p(0) = p_0$. Find solution $p(t)$. 

Compared with above: \(k, r\) play the roles of \(a, b\), respectively. So solution is \[ p(t) = \frac{r}{k} + \left(p_0 - \frac{r}{k}\right)e^{kt} \]

How does \(p(t)\) behave as \(t\) increases? Your qualitative description will depend on \(p_0\).

(i) \(p_0 = \frac{r}{k}\): \(p(t) = \frac{r}{k}\) (equilibrium solution).

(ii) \(p_0 > \frac{r}{k}\): \(p(t)\) increases without a bound (exponentially).

(iii) \(p_0 < \frac{r}{k}\): \(p(t)\) decreases, and eventually becomes 0 at the extinction time \(t_0\), which satisfies \[ p(t_0) = 0 = \frac{r}{k} + \left(p_0 - \frac{r}{k}\right)e^{kt_0} \]. Solve for \(t_0\):
\[ e^{kt_0} = \frac{r}{k} - \frac{r}{k - p_0} \]
Thus \(t_0 = -\frac{1}{k} \ln \frac{r}{k - p_0}\).

### 2. Free fall
\[
\frac{dv}{dt} = g - \frac{\gamma}{m} v, \quad v(0) = v_0.
\]
Here \(a = -\frac{\gamma}{m}, \quad b = -g\).

\[
v(t) = \frac{mg}{\gamma} \left( v_0 - \frac{mg}{\gamma} \right) e^{-\gamma t/m}
\]
Note: no matter what \(v_0\) is, we have \(\lim_{t \to \infty} v(t) = \frac{mg}{\gamma}\) (terminal velocity).

**Example.** A body with mass of 10kg is dropped from altitude 1000m, at \(t = 0\). The air drag \(\gamma = 2\text{kg/s}\). What is the terminal velocity? When will this body hit the ground, and at what speed?

We have \(\frac{dv}{dt} = 9.8 - \frac{2v}{10}\), so \(a = -1/5, \quad b = -9.8\). Initial condition: \(v(0) = 0\). Solution
\[
v(t) = 49 - 49e^{-t/5}.
\]
Terminal velocity is \(\lim_{t \to \infty} v(t) = 49\text{m/s}\).

Next solve for position \(x(t)\) (which is measured downward). Choose the origin of the coordinate system at the dropping point, so \(x(0) = 0\) and the ground impact happens when \(x(t) = 1000\).

We have \(\frac{dx}{dt} = v = 49 - 49e^{-t/5}\). Integrate: \(x = 49t + 245e^{-t/5} + C\). To compute \(C\) recall our initial condition: \(x(0) = 0 = 245 + C\). Thus, \(C = -245\), which yields
\[
x(t) = 49t - 245\left(1 - e^{-t/5}\right).
\]
To find the time of impact, solve \(x(t) = 1000\) (numerically with graphing calculator — it cannot be solved algebraically). Obtain \(t \approx 25.4\). Then \(v(25.4) = 49\left(1 - e^{-25.4/5}\right) \approx 48.7\text{m/s}\).

**Special case \((b = 0)\)**

DE \[
\frac{dy}{dt} = ay\] has solution \(y = Ce^{at}\) (exponential growth/decay with rate \(a\))

derivative = multiple of function \((e.g., \frac{dy}{dx} = 7y, \quad \frac{dq}{dt} = 7q, \quad \frac{dw}{dz} = 7w\) are all the same DE, growth rate 7)