LECTURE 6 (Jan 28): Separable equations (Section 2.2)

In-class exercises for separable DEs.

(a) \( \frac{dy}{dx} = (y - 2) \cos x \). Note \( y(x) = 2 \) is a constant solution (VERIFY!). Separate variables to find other solutions.

(b) Solve \( \frac{dy}{dx} = \frac{e^x - 2}{2e^y + 1} \) with initial condition \( y(0) = 1 \).

(c) \( \frac{dy}{dx} = \frac{5x^4 - 4x + 1}{2(y + 2)} \), \( y(0) = -1 \). Solve for \( y \).

Example (Example 4, Section 2.3). A body of mass \( m \) is thrown upward, with speed \( v_0 \). How does the velocity \( v \) depend on the altitude \( x \)? What is the maximum altitude attained? What is the escape velocity – that is, the smallest value of \( v_0 \) for which the body does not return to Earth?

(a) First build a model: an ODE for \( v \).

Answer. Assume no air resistance. Let \( x \)-direction be upward from surface, \( R = \) Earth’s radius, \( v = \frac{dx}{dt} \).

Gravitational force at height \( x \) is \( -mg \frac{R^2}{(R + x)^2} \) (by inverse square law from center of Earth, assumed spherical).

Newton’s law: \( m \frac{dv}{dt} = -mg \frac{R^2}{(R + x)^2} \), hence \( \frac{dv}{dt} = -g \frac{R^2}{(R + x)^2} \).

But we want \( \frac{dv}{dx} \). Substituting Chain Rule \( \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} v \) gives

\[ v \frac{dv}{dx} = -g \frac{R^2}{(R + x)^2}, \quad v(0) = v_0. \]
(b) Solve for $v(x)$. Find the maximal value of the altitude $x$ (in terms of $v_0$).

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**Section 2.4. Existence and uniqueness of solutions.**

1. **Existence and uniqueness for LINEAR 1st order DE.**

Consider the linear initial value problem $y' + p(t)y = q(t), \ y(t_0) = y_0$ in standard form.

(i) Does this DE have a solution on some interval $(\alpha, \beta)$ that contains $t_0$? In other words: is there a solution curve through $(t_0, y_0)$, and how far does this curve extend?

(ii) If the solution exists, is it unique? In other words: is there only one solution curve through $(t_0, y_0)$?

**Theorem.** If $p$ and $q$ are continuous on an open interval $(\alpha, \beta)$, with $t_0 \in (\alpha, \beta)$, then our initial value problem has a unique solution on $(\alpha, \beta)$.

Reason: integrating factor $\mu(t) = e^{\int p(t) \, dt}$ gives solution $y = \frac{1}{\mu(t)} \left( \int \mu(t)q(t) \, dt + C \right)$. Textbook has full proof.

**Example.** Consider the linear initial value problem (IVP) $ty' + 2y = \frac{3}{t}, \ y(1) = 2$. Find an interval on which the solution is guaranteed to exist and be unique.

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**In-class exercise.** Solve this initial value problem.