There are five problems worth a total of 40 points.

Show your work. Circle your answers.

You may use that for $x$ near 1 we have $\ln(x) = (x - 1) - \frac{(x - 1)^2}{2} + \text{H.O.T.}$

You must not communicate with other students during this test.

No books, notes, calculators, or electronic devices allowed.

The exam ends promptly after 50 minutes.

Do not turn this page until instructed to.

There are several different versions of this exam.

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Circle your small section number

<table>
<thead>
<tr>
<th>Alex Duda</th>
<th>BD0 11am</th>
<th>Alok Tiwari</th>
<th>CD1 8am</th>
</tr>
</thead>
<tbody>
<tr>
<td>S. Ahlgren</td>
<td>BD1 9am</td>
<td>Alok Tiwari</td>
<td>CD2 10 am</td>
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<tr>
<td>Eunmi Kim</td>
<td>BD2 10am</td>
<td>M. Tip Phaovibul</td>
<td>CD3 11am</td>
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<tr>
<td>Eunmi Kim</td>
<td>BD3 11am</td>
<td>Chris Stocker</td>
<td>CD4 2pm</td>
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<tr>
<td>Brian Schertz</td>
<td>BD4 12pm</td>
<td>Sujana Chandrasekar</td>
<td>CD5 1pm</td>
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<td>Brian Schertz</td>
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<td>Jan Vervoort</td>
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<td>Jan Vervoort</td>
<td>BD6 3pm</td>
<td>Sujana Chandrasekar</td>
<td>CD7 3pm</td>
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<tr>
<td>Alex Duda</td>
<td>BD7 1pm</td>
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<td>Vyron Vellis</td>
<td>BD8 3pm</td>
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<tr>
<td>M. Tip Phaovibul</td>
<td>BD9 12pm</td>
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Do not write below this line—for graders only

<table>
<thead>
<tr>
<th>1 (8 points)</th>
<th>2 (8 points)</th>
<th>3 (8 points)</th>
<th>4 (6 points)</th>
<th>5 (10 points)</th>
<th>Total (40 points)</th>
</tr>
</thead>
</table>
1. (8 points) You are given the following table of values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>-3</td>
<td>3</td>
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<td>2</td>
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<tr>
<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

a) Let $g(x) = (f(x^3))^2$. Find $g'(-1)$.

$$
g'(x) = 2f(x^3)f'(x^3)3x^2 \quad \text{by Chain Rule}
$$

$$
g'(-1) = 2f(-1)f'(-1)3\cdot1
$$

$$
= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 1
$$

$$
= 36
$$

b) Let $h(x) = f(f(\cos x))$. Find $h'(\pi/2)$.

$$
h'(x) = f'(f(\cos x))f'(\cos x)(-\sin x) \quad \text{by Chain Rule}
$$

$$
h'\left(\frac{\pi}{2}\right) = f'(f(0))f'(0)(-1)
$$

$$
= f'(3)f'(0)(-1)
$$

$$
= (-1)(-3)(-1)
$$

$$
= -3
$$
2. (8 points) Use the precise definition of limit (in other words, the \( \varepsilon-\delta \) definition) to show that

\[
\lim_{x \to 0} \frac{(2+3x)^2 - 4}{x} = 12.
\]

\[
\begin{align*}
\text{Want} & \quad \left| \frac{(2+3x)^2 - 4}{x} - 12 \right| < \varepsilon \\
\left| \frac{4 + 12x + 9x^2 - 4}{x} - 12 \right| & < \varepsilon \\
\left| 12 + 9x - 12 \right| & < \varepsilon \\
9|x| & < \varepsilon \\
|x| & < \frac{\varepsilon}{9}
\end{align*}
\]

So choose \( \delta = \frac{\varepsilon}{9} \).
3. (8 points) Let

\[ f(x) = \frac{1}{x} \frac{e^{4x} - e^{-2x}}{e^{4x} + e^{-5x}}. \]

a) Use series to evaluate \( \lim_{x \to 0} f(x) \).

\[
\begin{align*}
\frac{f(x)}{x} &= \frac{(1 + 4x + HoT) - (1 - 2x + HoT)}{(1 + 4x + HoT) + (1 - 5x + HoT)} \\
&= \frac{1}{x} \frac{6x + HoT}{2 - x + HoT} = \frac{6 + HoT}{2 - x + HoT} \\
\lim_{x \to 0} f(x) &= \frac{6}{2} = 3
\end{align*}
\]

b) Evaluate \( \lim_{x \to \infty} f(x) \) (provide brief but complete justification).

\[
\begin{align*}
f(x) &= \frac{1}{x} \frac{e^{4x} - e^{-2x}}{e^{4x} + e^{-5x}} \\
&= \frac{1}{x} \frac{1 - e^{-6x}}{1 + e^{-9x}} \\
\lim_{x \to \infty} f(x) &= 0 \\
\lim_{x \to \infty} & = 1
\end{align*}
\]

\[
\Rightarrow \lim_{x \to \infty} f(x) = 0 \cdot 1 = 0
\]
4. (6 points) A planet has a circular orbit around a sun. The period of the orbit is

\[ P = C \sqrt{\frac{R^3}{M}} \]

where \( C \) is a constant, \( R \) is the radius of the orbit, and \( M \) is the mass of the sun. The period decreases by 6% and the mass of the sun remains constant. Use differentials to estimate the percentage change in the radius.

\[ \ln P = \ln C + \frac{3}{2} \ln R - \frac{1}{2} \ln M \]

\[ \frac{dP}{P} = \frac{3}{2} \frac{dR}{R} - \frac{1}{2} \frac{dM}{M} \]

\[-0.06 = \frac{3}{2} \frac{dR}{R} \]

\[ \frac{dR}{R} = -0.06 \left( \frac{2}{3} \right) = -0.04 \]

\[ R \text{ decreases by about } 4\% \]
5. (10 points) A sphere of radius 1 and mass \( m \) moves horizontally through water. If the initial velocity is \( v_0 \) and the initial position is \( s_0 \), then the position at time \( t \) is given by

\[
s(t) = s_0 + \frac{m}{k} \ln \left(1 + \frac{k v_0}{m} t \right),
\]

where \( s_0, v_0, k \) are positive constants (\( k \) relates to the water resistance).

*Do not include units. Each brief interpretation should be one complete sentence only.*

a) Compute the velocity \( v(t) \) and simplify.

\[
V(t) = s'(t) = \frac{m}{k^2} \frac{k v_0}{1 + \frac{k v_0}{m} t} = \frac{V_0}{1 + \frac{k v_0}{m} t}
\]

b) Evaluate \( \lim_{t \to \infty} v(t) \) and give a brief physical interpretation.

\[= 0\]

As time becomes large, velocity \( \to 0 \).

c) Evaluate \( \lim_{t \to \infty} s(t) \) and give a brief physical interpretation.

\[= \infty\]

The sphere travels substantially far.

d) For a fixed time \( t \), evaluate \( \lim_{m \to \infty} v \). Give a brief physical interpretation.

\[\lim_{m \to \infty} v = \frac{V_0}{1 + 0} = V_0\]

Water resistance is negligible for large masses.

e) For a fixed time \( t \), evaluate \( \lim_{m \to \infty} s \).

When \( m \) large, \( 1 + \frac{k v_0}{m} t \) near 1.

\[
S(t) = S_0 + \frac{m}{k} \left( \frac{k v_0}{m} t + \text{HT} \right)
\]

\[
S(t) \to s_0 + v_0 t \quad \text{as} \quad m \to \infty.
\]